Published by Institute of Physics Publishing for SISSA
Received: *October 12, 2007*
Accepted: *November 20, 2007*
Published: *December 6, 2007*

From fake supergravity to superstars

Henriette Elvang,[b] **Daniel Z. Freedman**[ab] **and Hong Liu**[b]

[a] *Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, 2-381 Cambridge, MA 02139, U.S.A.*
[b] *Center for Theoretical Physics, Massachusetts Institute of Technology, 77 Massachusetts Ave, 6-304 Cambridge, MA 02139, U.S.A.*
E-mail: elvang@lns.mit.edu, dzf@math.mit.edu, hong_liu@mit.edu

Abstract: The fake supergravity method is applied to 5-dimensional asymptotically AdS spacetimes containing gravity coupled to a real scalar and an abelian gauge field. The motivation is to obtain bulk solutions with $\mathbb{R}\times S^3$ symmetry in order to explore the AdS/CFT correspondence when the boundary gauge theory is on $\mathbb{R}\times S^3$. A fake supergravity action, invariant under local supersymmetry through linear order in fermion fields, is obtained. The gauge field makes things more restrictive than in previous applications of fake supergravity which allowed quite general scalar potentials. Here the superpotential must take the form $W(\phi) \sim \exp(-k\phi) + c\ \exp(\frac{2\phi}{3k})$, and the only freedom is the choice of the constant k. The fermion transformation rules of fake supergravity lead to fake Killing spinor equations. From their integrability conditions, we obtain first order differential equations which we solve analytically to find singular electrically charged solutions of the Lagrangian field equations. A Schwarzschild mass term can be added to produce a horizon which shields the singularity. The solutions, which include "superstars", turn out to be known in the literature. We compute their holographic parameters.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetry and Duality, Supergravity Models.

JHEP12(2007)023

Contents

1. Introduction **1**

2. Basics of fake supergravity **5**
2.1 Structure of real and fake supergravity 5
2.2 Flat sliced domain walls 7
2.3 AdS_d sliced domain walls 8
2.4 $\mathbb{R} \times S^3$ solutions 9

3. Fake supergravity with a gauge field **9**
3.1 Fake supergravity action 9
3.2 Equations of motion 11
3.3 A c-theorem 12

4. Fake BPS and non-extremal solutions **13**
4.1 Integrability conditions for fake Killing spinors 13
4.2 Construction 14
4.3 Non-extremal solutions 17
4.4 Comparison with known solutions 17
4.5 Scalar "flow" and horizons 18

5. Mass and charge from holography **20**
5.1 Holographic stress-energy tensor 20
5.2 Witten-Nester in fake supergravity 23

6. Discussion **25**

A. Details of the fake supergravity construction **26**

B. Conditions for existence of a horizon **29**

C. Fake Killing spinors **30**

1. Introduction

The AdS/CFT correspondence has stimulated the study of asymptotically anti-de Sitter spacetimes in various dimensions. Quite often these spacetimes are solutions of a supergravity theory containing gravity coupled to bosonic matter fields. In this setting, it is common to search first for BPS solutions which support Killing spinors. The BPS conditions are first order differential equations which are frequently easier to solve than the Lagrangian field equations. BPS solutions have residual supersymmetry. They are a small subset of the solutions one would like to study.

The purpose of the fake supergravity method is to obtain workable first order equations whose solutions also satisfy the Lagrangian equations of motion, but are applicable to non-BPS solutions of true supergravity theories and to theories which have only a rough resemblence to supergravity. Even the limitation to spacetime dimension $D \leq 11$ can be overcome in this framework. The method proceeds by formulating fake Killing spinor equations whose integrability conditions are the needed first order equations. One can then attempt to solve these equations to find new spacetimes or, in combination with the Witten-Nester approach to gravitational stability, use it to establish linear stability of previously known non-BPS solutions.

This approach was first devised for flat-sliced domain walls in [1] and [2] for a bosonic action of the form

$$S_{\rm B} = \int d^{d+1}x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{1.1}$$

The metric and scalar field of these domain walls take the form

$$\begin{aligned} ds^2_{d+1} &= e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2\,, \\ \phi &= \phi(r)\,. \end{aligned} \tag{1.2}$$

The warp factor e^{2A} multiplies the metric of d-dimensional Minkowski spacetime. The basic quantity in the fake supergravity method is the real superpotential $W(\phi)$ which is related to the scalar potential by

$$V(\phi) = 2(d-1)^2\left[W'(\phi)^2 - \frac{d}{d-1}W(\phi)^2\right]. \tag{1.3}$$

The first order flow equations obtained in [1, 2], namely

$$\begin{aligned} \phi'(r) &= -2(d-1)W'(\phi)\,, \\ A'(r) &= 2W(\phi(r))\,, \end{aligned} \tag{1.4}$$

were later shown in [3] to be Hamilton-Jacobi equations for the domain wall dynamics obtained from the field equations of (1.1), in which $W(\phi)$ is Hamilton's principal function. The fake supergravity (or Hamilton-Jacobi) method has had several applications, especially to brane-world models [2, 4, 5] with stabilized inter-brane spacing.

The fake supergravity method works, and is far less restrictive than true supergravity, because it requires the general structure of supergravity only to lowest order in fermion fields. Specifically, as we show in section 2, one can find a fermion action $S_{\rm F}$, strictly bilinear in the gravitino and dilatino fields ψ_μ and λ, such that the sum $S_{\rm B} + S_{\rm F}$ is invariant under local supersymmetry, but only to linear order in ψ_μ and λ. To this order, one requires detailed γ-matrix algebra, but dimension-specific properties such as Fierz rearrangement

JHEP12(2007)023

are not used. The fake Killing spinor equations are the conditions $\delta\psi_\mu = 0$ and $\delta\lambda = 0$ obtained from the fermion variations used to demonstrate linear local supersymmetry.

The next step in the development was the extension of the method to AdS_d sliced domain walls [6]. The new metric ansatz replaces the Minkowski metric $\eta_{\mu\nu}$ in (1.2) with an AdS_d metric $g_{\mu\nu}$. The fake supergravity framework for flat-sliced walls must be modified because the Lagrangian equations of motion change. The needed modification incorporates a feature of true $D = 5$, $\mathcal{N} = 2$ supergravity, namely that the scalar superpotential W is replaced by an SU(2) matrix[1] $\mathbf{W}$ subject to a further constraint reviewed in section 2.3. This modification was applied [6] to the stability problem of the Janus solution [8] of $D = 10$ Type IIB supergravity. The structure of fake supergravity was further studied in [9].

In this paper we extend the fake supergravity method to $\mathbb{R} \times S^3$-sliced domain walls in a 5-dimensional bulk. Our motivation is to explore the AdS/CFT correspondence for the situation of the boundary gauge theory on $\mathbb{R} \times S^3$. Many recent applications of AdS/CFT involve the $D = 4$ $\mathcal{N} = 4$ SYM theory on this manifold.

The bosonic action which governs our system is

$$S_{\mathrm{B}} = \int d^5x \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}Q(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)\right]. \tag{1.5}$$

It includes an abelian gauge field A_μ with non-minimal coupling to the scalar ϕ. In section 3 we construct actions S_{F} and S_{gauge} quadratic in the gravitino and dilatino fields ψ_μ and λ such that the total action $S = S_{\mathrm{B}} + S_{\mathrm{F}} + S_{\mathrm{gauge}}$ is invariant to linear order in the fermions under local supersymmetry transformations. These are motivated by the structure of real 5D supergravity. A main consequence of linear local supersymmetry is that the function Q and the superpotential W are required to take the specific form

$$Q(\phi) = e^{2k\phi}\,, \qquad W(\phi) = w_1\, e^{-k\phi} + w_2\, e^{\frac{2}{3k}\phi}\,, \tag{1.6}$$

where w_i are constants of integrations. Thus the only freedom is the choice of the constant k. This is quite different from the previously studied fake supersymmetric actions which admitted arbitrary superpotentials in the absence of the gauge field. The scalar potential resulting from W in (1.6) via (1.3) has a local maximum. Scalar fluctuations around this local maximum have mass^2 saturating the Breitenlohner-Freedman (BF) bound [10] for all k. The bulk scalar ϕ approaches the local maximum at the AdS boundary of all our solutions, and is therefore dual to a $\Delta = 2$ boundary gauge theory operator.

For $\mathbb{R} \times S^3$-slicings the gauge field is necessary for non-trivial solutions of the first order equations. We impose a static solution ansatz which preserves spherical symmetry and includes only an electric component of the gauge field. This leaves four functions to be solved for as functions of a radial coordinate r: the scalar $\phi(r)$ field, the gauge potential $A_t = a(r)$, and two functions $A(r)$ and $B(r)$ which are warp factors in the metric. The fake supersymmetry transformations of the gravitino and dilatino yield fake Killing spinor conditions $\delta\psi_\mu = 0$ and $\delta\lambda = 0$. Their integrability conditions give rise to first order "flow" equations for the four functions A, B, ϕ, a of our ansatz (section 4.1).

[1] A formulation in terms of the Hamilton-Jacobi equations was given in [7].

JHEP12(2007)023

The first order equations can be solved analytically; we show how in section 4. The solutions of the flow equations are fake BPS in the sense that they admit fake Killing spinors. The electrically charged solutions are all nakedly singular, but a non-extremality parameter μ can be introduced to hide the singularity behind an event horizon. The general non-extremal solutions can then be written

$$\begin{aligned} ds_5^2 &= -H^{-2p}\, f\, dt^2 + H^p\,(f^{-1}\,dy^2 + y^2 d\Omega_3^2) \\ A_t &= -\frac{\tilde{q}}{q}\sqrt{\frac{3}{2+3k^2}}\,(H^{-1}-1) \\ e^{\frac{2}{3k}\phi} &= H^p\,, \end{aligned} \tag{1.7}$$

with

$$H(y) = 1 + \frac{q}{y^2}\,, \qquad f(y) = 1 + \frac{y^2}{L^2}H^{3p} - \frac{\mu}{y^2} \tag{1.8}$$

and

$$p = \frac{2}{2+3k^2}\,, \qquad \tilde{q}^2 = q(q+\mu)\,. \tag{1.9}$$

Asymptotically ($y \to \infty$) the solutions approach global AdS_5. The parameter $\tilde{q}$ is proportional to the electric charge.

Various special cases of the solutions (1.7) had previously appeared in the literature [11−15]. Most notably, the fake BPS solutions (which have $\mu = 0$) are truly supersymmetric for the special values of $k = 0,\ 1/\sqrt{3},\ 2/\sqrt{3}$ as they then arise as solutions of consistently truncated $D = 5$, $\mathcal{N} = 2$ gauged $\mathrm{U}(1)^3$ supergravity theory (see for instance [16]). The Schwarzschild type mass term μ was added in [13] providing a horizon, and thus giving regular non-BPS charged spherically symmetric black holes. (In AdS_5, regular BPS black holes carry non-vanishing angular momentum [17].) The type IIB lift of the solutions was given in [15, 18], and later interpreted in [14] as "superstars" describing continuous distributions of giant gravitons.

After we found the solutions (1.7) for general k, we learned that they had been constructed earlier by Gao and Zhang [19] who worked with the second order Lagrangian field equations. Fake supergravity gives some insight into the structure of the scalar potential. Here we also analyze bulk and AdS/CFT properties of the solutions.

In the context of the AdS/CFT correspondence, previously studied domain wall solutions and their generalizations had interpretations as gravitational duals of renormalization group flows, and a holographic c-theorem was derived [20, 21]. Motivated by this, we construct here a c-function which is monotonically decreasing as the scalar flows from the asymptotic AdS boundary to the interior. This result relies only on the structure of the field equations.

We compute in section 5 the holographic stress tensor from which we derive the mass of the system. All fields of the solutions (1.7) approach the boundary at their "vev rate". For $q > 0$, the extremal solution should be the gravity dual of an excited state of the boundary gauge theory with non-vanishing charge and vev for a scalar operator with $\Delta = 2$. When the solutions have horizons, we have the dual of an ensemble of such states at fixed temperature. Since the gauge theory is on the compact domain S^3, the charge is that of a global symmetry. For the k-values in which the solutions coincide with the superstars of [14], this is an SO(2) subgroup of the SO(6) R-symmetry of $d = 4$ $\mathcal{N} = 4$ SYM theory.

The mass obtained from the holographic stress tensor is suggestive of a BPS bound saturated for the $\mu = 0$ solutions. We compute the Witten-Nester energy for all solutions, but find that despite the existence of fake BPS Killing spinors there is an obstruction to deriving a fully general BPS bound for all k. Restricting to the class of solutions for which $F \wedge F$ vanishes, however, allow us to confirm the bound suggested by the holographic mass calculation.

Section 6 contains a brief discussion. The main paper is concerned with fake supergravity in $D = 5$, but appendix A provides details of the derivation of linear supersymmetry for general dimensions $D = d + 1 \geq 4$. For all k, the bulk scalar is dual to a putative boundary theory operator of dimension $\Delta = d - 2$, which is the dimension of a scalar mass operator. Appendix B analyzes conditions for the existence of horizons, and appendix C constructs fake Killing spinors for the fake BPS solutions.

2. Basics of fake supergravity

We introduce the basic structure of fake supergravity and present as examples the construction of flat- and AdS-sliced domain wall solutions.

2.1 Structure of real and fake supergravity

Fake supergravity shares the structure of the Lagrangian and transformation rules of supergravity, but requires local supersymmetry only to linear order in fermion fields. Linear local supersymmetry allows more freedom in the bosonic sector, even the freedom of arbitrary spacetime dimension.

To see why this works, consider a generic true supergravity theory with a collection of boson and fermion fields $B(x)$ and $F(x)$ and transformation rules which involve arbitrary spinor parameters $\epsilon(x)$. The action $S[B, F]$ is locally supersymmetric, which means that the supersymmetry variation

$$\delta S = \int d^D x \left(\frac{\delta \mathcal{L}}{\delta B} \delta B + \frac{\delta \mathcal{L}}{\delta F} \delta F \right) \equiv 0 \tag{2.1}$$

vanishes *identically*, for *all* configurations of $B(x), F(x), \epsilon(x)$. In particular, the terms of each order in F vanish *independently*. To lowest order, with fermions more specifically described as gravitinos $\psi_\mu(x)$ and Dirac spinors $\lambda(x)$, the fermion transformations have the generic structure

$$(\delta B)_0 = \bar{\epsilon} \Gamma F \ = \ \bar{\epsilon} \left(\Gamma^\mu \psi_\mu + \Gamma' B \lambda \right) \tag{2.2}$$

$$(\delta F)_0 = \begin{cases} (\delta \psi_\mu)_0 = (D_\mu + \Gamma''{}_\mu B) \epsilon \\ (\delta \lambda)_0 = (\Gamma^\mu \partial_\mu B + \Gamma''' B) \epsilon . \end{cases} \tag{2.3}$$

The Γ, Γ', etc. are matrices of the Clifford algebra with the appropriate tensor structure.

The lowest order term in δS is *linear* in the fermions; it takes the form

$$(\delta S)_{\rm lin} = \int d^D x \left[\frac{\delta \mathcal{L}}{\delta B}(\bar\epsilon \Gamma F) + \frac{\delta \mathcal{L}}{\delta F}(\delta F)_0\right] \equiv 0. \tag{2.4}$$

The variation $\delta\mathcal{L}/\delta B$ is purely bosonic to this order, and $\delta\mathcal{L}/\delta F$ is linear in fermions. Note that $(\delta S)_{\rm lin}$ still vanishes for all configurations of $B(x), F(x), \epsilon(x)$. If ϵ is a Killing spinor, then, by definition $(\delta F)_0 = 0$, and (2.4) then reads

$$(\delta S)_{\rm lin} = \int d^D x\, \frac{\delta \mathcal{L}}{\delta B}(\bar\epsilon \Gamma F) \;=\; 0. \tag{2.5}$$

It vanishes for all configurations of $B(x)$ which support Killing spinors and all fermion configurations $F(x)$. Thus the sum over all independent boson fields $B_I(x)$ vanishes locally, viz.

$$\sum_I \frac{\delta \mathcal{L}}{\delta B_I}(\bar\epsilon \Gamma F)_I = 0. \tag{2.6}$$

If the fermion variations $(\bar\epsilon\Gamma F)_I$ are independent, then each boson equation of motion $\delta\mathcal{L}/\delta B_I = 0$ is satisfied separately.

In many cases the fermion variations are independent, in other cases one must supplement the equations (2.6) with gauge field equations of motion [22]. It is in this way that a bosonic field configuration $B_I(x)$ which supports Killing spinors can give a solution of the bosonic equations of motion of the theory. The first order equations which determine these BPS configurations of $B_I(x)$ are the integrability conditions for the Killing spinor equations $(\delta\psi_\mu)_0 = 0$ and $(\delta\lambda)_0 = 0$.

To see more specifically how fake supergravity imitates and extends this result, we construct the linear supergravity for the bosonic action

$$S_{\rm B} = \int d^D x \sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{2.7}$$

We do this in some detail because the construction seems to be new and is an independent sector of the more general situation with gauge field. We consider the action $S = S_{\rm B} + S_{\rm F}$ where $S_{\rm F}$ is strictly bilinear in the supersymmetry partners ψ_μ and λ of the bosons. $S_{\rm F}$ contains all fermion bilinears suggested by true supergravity, each with an unknown function of ϕ as coefficient, viz.

$$\begin{aligned} S_{\rm F} = \int d^D x\, \sqrt{-g}\,\Big[&4\,\bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho + \bar\lambda\Gamma^\mu D_\mu\lambda - A(\phi)\,\bar\lambda\lambda - B(\phi)\,\bar\psi_\mu\Gamma^{\mu\nu}\psi_\nu \\ &-\partial_\nu\phi\,(\bar\psi_\mu\Gamma^\nu\Gamma^\mu\lambda - \bar\lambda\,\Gamma^\mu\Gamma^\nu\,\psi_\mu) - C(\phi)(\bar\psi_\mu\Gamma^\mu\lambda - \bar\lambda\,\Gamma^\mu\psi_\mu)\Big]\,. \end{aligned} \tag{2.8}$$

The accompanying linearized transformation rules are

$$\begin{aligned} \delta\psi_\mu &= \big(D_\mu + \Gamma_\mu W(\phi)\big)\epsilon\,, & \delta\lambda &= \big(\Gamma^\mu\partial_\mu\phi - E(\phi)\big)\epsilon\,, \\ \delta e_\mu^a &= -2\big(\bar\epsilon\gamma^a\psi_\mu - \bar\psi_\mu\gamma^a\epsilon\big)\,, & \delta\phi &= -\bar\epsilon\lambda - \bar\lambda\epsilon\,. \end{aligned} \tag{2.9}$$

The derivative D_μ includes the spin-connection,

$$D_\mu = \nabla_\mu \equiv \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\,. \tag{2.10}$$

A note on conventions: we use upper case Γ for gamma-matrices with coordinate indices, and lower case γ for gamma-matrices with frame indices. We use $\bar{\psi} = i\psi^\dagger\gamma^t$.

With some work,[2] one can show that the local supersymmetry variation $\delta(S_{\rm B} + S_{\rm F})$ vanishes, provided that the potential $V(\phi)$ is related to the superpotential $W(\phi)$ by (1.3), and the unspecified functions of the ansatz (2.8)-(2.9) are given by

$$A = -(d-1)\big(2W'' - W\big)\,, \qquad B = 4(d-1)W\,, \qquad C = E = 2(d-1)W'\,. \tag{2.11}$$

The computations needed to prove linear local supersymmetry are similar to those of the component approach to supergravity. They require considerable γ-matrix algebra, but dimension specific manipulations, such as Fierz rearrangement, are not required at linear order. This is the reason that linear local supersymmetry is valid for any dimension! One further difference is that in fake supergravity it is not necessary to specify the class of spinor, e.g. symplectic Majorana spinors in true $D = 5$ supergravity. In our computations we assume that all spinors are complex Dirac spinors.

The derivation of the linear fake supergravity action and transformation rules depends only on the bosonic fields one is working with, in this case the metric $g_{\mu\nu}$ and a single scalar ϕ. It does not depend on the symmetries of the solution which is sought. In the next stage of the program one uses the fermion transformation rules of (2.9) as fake Killing spinor equations and explores their integrability conditions in spacetimes of specific symmetry, such as flat- and AdS-sliced domain walls. These examples are described briefly below.

The construction of the linear local supersymmetry theory can be bypassed as was done for flat- or AdS-sliced domain walls in [1, 2, 6]. In the more complicated case of $\mathbb{R}\times S^3$ slicing, we found it useful to work in two stages, first to construct the linear supergravity model and then study the resulting fake Killing spinor conditions obtained from the model.

2.2 Flat sliced domain walls

The metric and scalar field of these domain walls take the form

$$\begin{aligned} ds^2_{d+1} &= e^{2A(r)}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2\,, \\ \phi &= \phi(r). \end{aligned} \tag{2.12}$$

The warp factor e^{2A} multiplies the metric of d-dimensional Minkowski spacetime. When the ansatz (2.12) is inserted in the fake Killing spinor conditions (see (2.9)-(2.11)), they reduce to

$$\mathcal{D}_r\epsilon = \delta\psi_r \;=\; \Big(\partial_r + \gamma_r W\Big)\epsilon \;=\; 0\,, \tag{2.13}$$

$$\mathcal{D}_i\epsilon = \delta\psi_i \;=\; \Big(\partial_i - \frac{1}{2}A'\gamma^i\gamma^r + \gamma^i\,W\Big)\epsilon \;=\; 0\,, \tag{2.14}$$

$$\hat{\mathcal{D}}\epsilon = \delta\lambda \;=\; \Big(\gamma^r\phi' - 2(d-1)W'\Big)\epsilon \;=\; 0\,. \tag{2.15}$$

[2]More specifically, one finds linear conditions relating the unknown functions by requiring that the coefficients of the following terms in $\delta(S_{\rm B}+S_{\rm F})$ each vanish: $(\bar{\lambda}\Gamma\cdot D\epsilon)$, $(\bar{\lambda}\Gamma\cdot\partial\phi\,\epsilon)$, $(\bar{\psi}_\mu\Gamma^{\mu\nu}D_\nu\epsilon)$, $(\bar{\psi}_\mu\partial^\mu\phi\epsilon)$ and $(\bar{\psi}_\mu\Gamma^{\mu\nu}\partial_\nu\phi\,\epsilon)$, and quadratic conditions from the coefficients of $(\bar{\lambda}\epsilon)$ and $(\bar{\psi}_\mu\Gamma^\mu\epsilon)$. The relation (1.3) appears as the coefficient of $(\bar{\psi}_\mu\Gamma^\mu\epsilon)$

The condition $\hat{\mathcal{D}}\epsilon = 0$ implies $\phi'^2 = 4(d-1)^2 W'^2$. Consistency of (2.13) and (2.15) requires

$$[\mathcal{D}_i, \hat{\mathcal{D}}]\epsilon = -\gamma^i \Big(A'\phi' - 2W\phi'\,\gamma^r \Big)\epsilon = 0\,, \tag{2.16}$$

which by (2.15) implies $A'\phi' = -4(d-1)WW'$. Choosing a definite sign for ϕ' we can now summarize the first order flow conditions

$$\begin{aligned}\phi'(r) &= -2(d-1)W'(\phi)\,,\\ A'(r) &= 2\,W\big(\phi(r)\big)\,.\end{aligned} \tag{2.17}$$

These equations are easily integrated and solve the field equations

$$d(d-1)\,A'^2 = \big(\phi'^2 - V(\phi)\big)\,, \tag{2.18}$$

$$\phi'' + d\,A'\phi' = \frac{\partial V}{\partial \phi}\,, \tag{2.19}$$

which are the independent equations obtained from the Einstein and scalar field equations within the ansatz (2.12). The relationship (1.3) between the potential V and the superpotential W is reproduced by (2.18) using (2.17). The Killing spinors take the form $\epsilon = e^{A/2}\eta$ where η is a constant spinor which satisfies $\gamma^r\eta = -\eta$.

2.3 AdS$_d$ sliced domain walls

The equations (2.18) are modified for the solution ansatz

$$ds^2_{d+1} = e^{2A(r)}g_{ij}(x)dx^\mu dx^\nu + dr^2, \tag{2.20}$$

where g_{ij} is an AdS$_d$ metric with scale L_d, by the addition of the term $-e^{-2A}/L_d^2$ on the right side of the A'^2 equation (2.18). The fake Killing conditions (2.14) are also modified, namely ∂_i is replaced by the AdS$_d$ covariant derivative $\nabla_i^{\mathrm{AdS}_d}$ in $\delta\psi_i$. Following the analysis of [6], one finds (from (4.11) of [6]) that the integrability conditions are *inconsistent*.

Although it is not obvious, the inconsistency can be cured by doubling the spinors and postulating a matrix superpotential $\mathbf{W} = \sigma^a\,W_a(\phi)$ (or equivalently a 3-vector W_a). The σ^a are the Pauli matrices. The matrix $\mathbf{W}$ must satisfy the commutator condition

$$\big[\mathbf{W}', (d-1)\mathbf{W}'' + \mathbf{W}\big] \;=\; 0. \tag{2.21}$$

If this condition is satisfied then the fake Killing conditions are consistent, and any solution of the flow equations

$$\phi' = 2(d-1)\sqrt{W_aW_a}\,, \tag{2.22}$$

$$e^{-2A} = 4L_d^2\frac{(W_aW_a)(W_b'W_b') - (W_aW_a')^2}{(W_b'W_b')}\,, \tag{2.23}$$

produces a solution of the Lagrangian equations of motion. Note that the second equation is algebraic. When W_a and W_a' are parallel vectors, the inconsistency referred to above is visible in (2.23). See [6] for further details of the analysis.

It was not necessary to construct a linear fake supergravity model in [6], but it is quite easy to do so as outlined above. After doubling all spinors and including $\mathbf{W}$ one finds that little new is required. The commutator condition (2.21) emerges as a condition for linear supersymmetry. The relation (1.3) between V and $\mathbf{W}$ changes only by the replacements $W^2 \to W_aW_a$ and $W'^2 \to W_b'W_b'$.

2.4 $\mathbb{R} \times S^3$ solutions

With $\mathbb{R} \times S^3$-slicing, the metric and scalar fields take the form

$$ds_5^2 = -e^{2A(r)}dt^2 + dr^2 + e^{2B(r)}d\Omega_3^2 , \qquad \phi = \phi(r) .$$

Because of the positive curvature of S^3, the Killing spinor equations (2.9)-(2.11) allow only pure AdS$_5$ as a solution, even with a matrix superpotential. To obtain more general solutions we add a gauge field to the system, as we discuss in the next section.

3. Fake supergravity with a gauge field

We first describe how to modify the fake supergravity action to include a gauge field coupled to the scalar, then study the equations of motion and a c-theorem.

3.1 Fake supergravity action

In this section we outline the construction of the fake supergravity model associated with the bosonic action

$$S_{\rm B} = \int d^Dx \sqrt{-g}\Big[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{4}Q(\phi)F_{\mu\nu}F^{\mu\nu} - V(\phi)\Big] . \qquad (3.1)$$

It is a considerable complication to add the gauge field to the previous model specified by (2.7), (2.8) and (2.9). We need to construct an additional bilinear fermion action $S_{\rm gauge}$ and transformation rules so that the variation of the total action

$$S = S_{\rm B} + S_{\rm F} + S_{\rm gauge} \qquad (3.2)$$

vanishes to linear order in ψ_μ and λ. $S_{\rm B}$ is given in (3.1) and $S_{\rm F}$ in (2.8), and we take for $S_{\rm gauge}$

$$\begin{aligned} S_{\rm gauge} = \int d^Dx\sqrt{-g}\,\Big[& -i\,M(\phi)\,\bar\psi_\mu\big(\Gamma^\mu\Gamma^{\rho\sigma}\Gamma^\nu - \Gamma^\nu\Gamma^{\rho\sigma}\Gamma^\mu\big)\psi_\nu F_{\rho\sigma} + i\,P(\phi)\,\bar\lambda\,\Gamma^{\rho\sigma}F_{\rho\sigma}\lambda \\ & +i\,N(\phi)\,\big(\bar\psi_\mu\Gamma^{\rho\sigma}\Gamma^\mu\lambda - \bar\lambda\,\Gamma^\mu\Gamma^{\rho\sigma}\psi_\mu\big)F_{\rho\sigma}\Big] . \end{aligned} \qquad (3.3)$$

Each term in $S_{\rm gauge}$ consists of a fermion bilinear with the same γ-matrix structure as in true supergravity multiplied by a function of ϕ to be determined. The gravitino ψ_μ and the dilatino λ are charged, hence the derivatives D_μ that appear in the two fermion kinetic terms of (2.8) now include a coupling to the gauge field,

$$D_\mu = \nabla_\mu + icA_\mu , \qquad (3.4)$$

where ∇_μ as defined in (2.10) contains the spin connection. Gauge invariance requires that the gravitino and dilatino carry the same charge so that the mixed $\bar\lambda(\cdots)\psi_\mu$ terms in (2.8) are gauge invariant. The scalar ϕ is neutral.

JHEP12(2007)023

We also postulate the following transformation rules

$$
\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu + \Gamma_\mu W(\phi) + iX(\phi)\big(\Gamma_\mu{}^{\nu\rho} - 2(D-3)\,\delta_\mu^\nu\,\Gamma^\rho\big)F_{\nu\rho} + icA_\mu\right]\epsilon\,, \\
\delta\lambda &= \left[\Gamma^\mu\,\partial_\mu\phi - 2(D-2)W'(\phi) + i\,Y(\phi)\,\Gamma^{\rho\sigma}F_{\rho\sigma}\right]\epsilon\,, \\
\delta e_\mu^a &= -2\big(\bar\epsilon\gamma^a\psi_\mu - \bar\psi_\mu\gamma^a\epsilon\big)\,, \\
\delta\phi &= -\bar\epsilon\lambda - \bar\lambda\epsilon\,, \\
\delta A_\mu &= -i\alpha(\phi)\big(\bar\epsilon\,\psi_\mu - \bar\psi_\mu\epsilon\big) - i\beta(\phi)\big(\bar\epsilon\,\Gamma_\mu\lambda + \bar\lambda\,\Gamma_\mu\epsilon\big)\,.
\end{aligned}
\tag{3.5}
$$

The requirement of linear local supersymmetry of the total action (3.2) fixes all unknown scalar functions and the U(1) coupling c. Terms independent of the gauge field are a closed sector of the calculation, so the results (2.11) for A, B, C, E remain valid. To extend linear local supersymmetry to the gauge sector, we need to examine about 16 distinct spinor bilinears. The coefficient of each is a combination of the unspecified scalar functions of the ansatz in (3.3) and (3.5) and derivatives of those functions. Each such combination must vanish. The information in these conditions fixes the scalar functions uniquely up to integration constants which we then specify by imposing physical normalization conditions. The analysis of the 16 conditions is tedious, so we simply quote results for $D = 5$ here. Further details for general D are given in appendix A.

The results for Q, X, Y, W, and c which are actually needed to study the fake Killing spinor conditions are:

$$
\begin{aligned}
&Q(\phi) = e^{2k\phi}\,, \qquad X(\phi) = \frac{1}{4\sqrt{3(2+3k^2)}}\,e^{k\phi}\,, \qquad Y(\phi) = \;6k\,X(\phi)\,, \\
&W(\phi) = \frac{1}{2L(2+3k^2)}\Big(2\,e^{-k\phi} + 3k^2\,e^{\frac{2\phi}{3k}}\Big)\,, \qquad c = \;-\frac{1}{L}\sqrt{\frac{3}{2+3k^2}}\,,
\end{aligned}
\tag{3.6}
$$

while scalar functions in the actions (2.8) and (3.3) and the boson transformation rules of (3.5) are

$$
\begin{aligned}
M(\phi) &= -6X(\phi)\,, \quad & N(\phi) &= Y(\phi)\,, \quad & P(\phi) &= 3(1-2k^2)X(\phi)\,, \\
\alpha(\phi) &= -24\,\frac{X(\phi)}{Q(\phi)}\,, \quad & \beta(\phi) &= 12k\,\frac{X(\phi)}{Q(\phi)}\,.
\end{aligned}
\tag{3.7}
$$

The potential $V(\phi)$, obtained by inserting the superpotential from (3.6) into (1.3), has a unique local maximum at $\phi = 0$. This is the asymptotic value of the scalar in all the solutions we obtain. It is straightforward to expand the potential near the maximum and compare with the potential of a general massive scalar in AdS_5 of scale L:

$$
V(\phi) = -\frac{6}{L^2} - \frac{2}{L^2}\phi^2 \;=\; -\frac{6}{L^2} + \frac{1}{2}m^2\phi^2. \tag{3.8}
$$

We see that the parameter k has cancelled and the bulk scalar field has mass, $m^2 = -4/L^2$, thus saturating the BF bound [10] for all values of k. It is curious that fake supergravity, with one scalar and one gauge field, selects this value.[3] The potential is analyzed further in section 4.5.

[3] As shown in appendix A, for $D \neq 5$ the mass m^2 is strictly above the BF bound.

3.2 Equations of motion

The goal of this paper is to apply fake supergravity methods to obtain solutions of the equations of motion of the bosonic theory (3.1) within the ansatz

$$
\begin{aligned}
ds_5^2 &= -e^{2A(r)}dt^2 + e^{2h(r)}dr^2 + e^{2B(r)}d\Omega_3^2\,, \\
\phi &= \phi(r)\,, \\
F_{rt} &= \partial_r A_t(r) \equiv a'(r)\,.
\end{aligned}
\tag{3.9}
$$

The gauge field configuration is purely electric. The function $h(r)$ specifies the choice of radial coordinate, and we keep this freedom because different coordinates are convenient at different points in our study. It is useful to employ a definite form of the 3-sphere metric, namely

$$
d\Omega_3^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2\,, \tag{3.10}
$$

with coordinate ranges $\theta \in [0, \frac{\pi}{2}]$ and $\phi, \psi \in [0, 2\pi]$.

The gravitational equations of motion are

$$
\begin{aligned}
R_{\mu\nu} &= T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T_\rho{}^\rho & (3.11)\\
&= \partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}\left(\frac{2}{3}V - \frac{1}{6}Q\,F^2\right) + Q\,F_\mu{}^\rho F_{\nu\rho}. & (3.12)
\end{aligned}
$$

In the ansatz (3.9), these equations become

$$
\begin{aligned}
R_{rr} &= -A'' - 3B'' + (A' + 3B')(h' - A' - B') + 4A'B' & (3.13)\\
&= \phi'^2 + \frac{2}{3}e^{2h}\,V - \frac{2}{3}e^{-2A}\,a'^2\,Q\,, & \\
R_{tt} &= e^{2A-2h}\big(A'' + A'(A' + 3B' - h')\big) & (3.14)\\
&= -\frac{2}{3}e^{2A}\,V + \frac{2}{3}e^{-2h}\,a'^2\,Q\,, & \\
R_{\theta\theta} &= 2 - e^{2B-2h}\big(B'' + B'(A' + 3B' - h')\big) & (3.15)\\
&= e^{2B}\left(\frac{2}{3}V + \frac{1}{3}e^{-2A-2h}\,a'^2\,Q\right). &
\end{aligned}
$$

Note that $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$, $R_{\psi\psi} = \cos^2\theta R_{\theta\theta}$, and that off-diagonal components of the Ricci and stress tensors vanish.

Later we will use the following combinations of the equations above:

$$
\begin{aligned}
e^{-A-3B+h}\left(e^{A+3B-h}\,B'\right)' &= -\frac{2}{3}e^{2h}\,V + 2e^{2h-2B} - \frac{1}{3}e^{-2A}\,a'^2\,Q\,, & (3.16)\\
e^{-A-3B+h}\left(e^{A+3B-h}\,A'\right)' &= -\frac{2}{3}e^{2h}\,V + \frac{2}{3}e^{-2A}\,a'^2\,Q\,, & (3.17)\\
\frac{1}{2}\phi'^2 - 3B'^2 - 3A'B' &= -3e^{2h-2B} + e^{2h}\,V + \frac{1}{2}e^{-2A}\,a'^2\,Q\,. & (3.18)
\end{aligned}
$$

The gauge field and scalar equations of motion are

$$\left(e^{-A+3B-h}\,Q\,a'\right)' = 0\,. \tag{3.19}$$

$$e^{-A-3B+h}\left(e^{A+3B-h}\,\phi'\right)' = e^{2h}\,\frac{\partial V}{\partial\phi} - \frac{1}{2}e^{-2A}\,a'^2\,\frac{\partial Q}{\partial\phi}\,. \tag{3.20}$$

The equations of motion can also be obtained from the one-dimensional effective action

$$S = -\int dr\, e^{A+3B-h}\left[\frac{1}{2}\phi'^2 - 3B'^2 - 3A'B' - \frac{1}{2}e^{-2A}\,a'^2\,Q(\phi) - 3e^{2h-2B} + e^{2h}\,V(\phi)\right]. \tag{3.21}$$

Note that the field equations are not all independent. For example, (3.17) can be derived by differentiating (3.18) and using the other equations of motion.

3.3 A c-theorem

The combination $R_{rr} + e^{-2(A-h)}R_{tt}$ of the Ricci tensor components gives

$$-3(B'' + B'^2 - A'B' - B'h') = \phi'^2. \tag{3.22}$$

If we choose $h = B - A$ and call the corresponding radial coordinate $\tilde{r}$, then, with $'$ denoting $d/d\tilde{r}$, we find that the quantity

$$C(\tilde{r}) \equiv \frac{\mathcal{C}_0}{B'(\tilde{r})^3}, \tag{3.23}$$

is monotonic increasing with $\tilde{r}$ for any positive constant $\mathcal{C}_0$.

We wish to adapt the argument of [20, 21] and interpret $C(\tilde{r})$ as a c-function. For this purpose we write the AdS_5 metric using two different radial coordinates, the first corresponds to $h = 0$ and the second is $\tilde{r}$:

$$ds^2_{\mathrm{AdS}_5} = -L^2\cosh^2\left(\frac{r}{L}\right)dt^2 + dr^2 + L^2\sinh^2\left(\frac{r}{L}\right)d\Omega_3^2 \tag{3.24}$$

$$= -L^2(1 + e^{2\tilde{r}/L})\,dt^2 + \frac{d\tilde{r}^2}{1 + e^{-2\tilde{r}/L}} + L^2\,e^{2\tilde{r}/L}\,d\Omega_3^2. \tag{3.25}$$

The two coordinates are related by $\tilde{r} = L\ln(\sinh(r/L))$. We see that $B'(\tilde{r}) = 1/L$. Now consider a solution of the equations of motion in which the $\mathbb{R}\times S^3$ sliced metric approaches the boundary region, $\tilde{r}\to+\infty$, of an AdS_5 spacetime with scale L_{UV} and the deep interior region, $\tilde{r}\to-\infty$, of an AdS_5 spacetime with scale L_{IR}. The c-function (3.23) then interpolates monotonically between these limits. With suitable normalization, i.e. choice of $\mathcal{C}_0$, it coincides at the endpoints with the central charge [23] of putative dual 4-dimensional conformal field theories on $\mathbb{R}\times S^3$. Since $L_{\mathrm{IR}} < L_{\mathrm{UV}}$, the central charge decreases in the renormalization group flow toward long distance.

It would be strange if the construction of a c-function required a particular radial coordinate, and indeed it does not. For any choice of $h(r)$, it is straightforward to see, using (3.22), that

$$C(r) \equiv \mathcal{C}_0\left(\frac{dB}{dr}\right)^{-3} e^{3(h+A-B)} \tag{3.26}$$

is monotonic and in fact coincides with $C(\tilde{r})$. The interpretation is more straightforward with the $\tilde{r}$ coordinate (and the AdS_5 warp factor $e^{\frac{2\tilde{r}}{L}}$ is pure exponential), but the physics is more general. The monotonicity of C depends only on the equations of motion for the solution ansatz, not the actual solution.

The interpretation of the c-function will be less clear for our solutions because they contain a singularity in the interior. It turns out that the c-function $C(r)$ is non-vanishing at horizons, when present, while $C(r)$ vanishes at the singularity.

4. Fake BPS and non-extremal solutions

We derive integrability conditions from the Killing spinor conditions of the fake supergravity model of section 3. We then solve these first order condition to obtain the most general fake BPS solutions within the ansatz (3.9). The solutions are then generalized to include a non-extremality parameter. We study relevant properties of the solutions.

4.1 Integrability conditions for fake Killing spinors

The Killing spinor equations obtained from the fermion transformation rules in (3.5) are

$$\begin{aligned} \mathcal{D}_\mu \epsilon &\equiv \Big[\nabla_\mu + iX(\phi)\Big(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu \Gamma^\rho\Big)F_{\nu\rho} + \Gamma_\mu W(\phi) + ic\,A_\mu\Big]\epsilon = 0\,, \\ \hat{\mathcal{D}}\epsilon &\equiv \Big[\Gamma^\mu\partial_\mu\phi - 6W'(\phi) + iY(\phi)\Gamma^{\mu\nu}F_{\mu\nu}\Big]\epsilon = 0\,. \end{aligned} \tag{4.1}$$

In the obvious diagonal frame for the metric (3.9)-(3.10), and with spin connections included, the operators in (4.1) become

$$\hat{\mathcal{D}} = e^{-h}\phi'\,\gamma^r - 6W' + 2ia'Ye^{-A-h}\gamma^r\gamma^t\,, \tag{4.2}$$

$$\mathcal{D}_t = \partial_t - \frac{1}{2}A'e^{A-h}\,\gamma^t\gamma^r + 4ia'Xe^{-h}\gamma^r - e^A W\gamma^t + icA_t\,, \tag{4.3}$$

$$\mathcal{D}_r = \partial_r + e^h W\,\gamma^r - 4ia'Xe^{-A}\,\gamma^t\,, \tag{4.4}$$

$$\mathcal{D}_\theta = \partial_\theta + \frac{1}{2}B'e^{B-h}\,\gamma^\theta\gamma^r + e^B W\,\gamma^\theta + 2ia'Xe^{B-A-h}\,\gamma^\theta\gamma^r\gamma^t\,, \tag{4.5}$$

$$\mathcal{D}_\phi = \partial_\phi + \frac{1}{2}\,\gamma^\phi\gamma^\theta\cos\theta + \gamma^\phi\left(\frac{1}{2}B'e^{B-h}\,\gamma^r + e^B W + 2ia'Xe^{B-A-h}\,\gamma^r\gamma^t\right)\sin\theta\,. \tag{4.6}$$

Note that $'$ means d/dr for the functions A, B, h, a, and ϕ of the solution ansatz of (3.9), but means $d/d\phi$ for the superpotential $W(\phi)$.

Fake Killing spinors are solutions of the equations (4.1). Solutions exist if the commutators of the 6 conditions vanish, i.e.

$$[\mathcal{D}_\mu, \hat{\mathcal{D}}]\epsilon = 0\,, \qquad [\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0\,. \tag{4.7}$$

The commutator conditions are a set of first and second order differential equations for $A(r)$, $B(r)$, $a(r)$, $\phi(r)$. It turns out that only the first order conditions, those obtained from commutators not involving $\mathcal{D}_r$, are sufficient to obtain solutions of the Lagrangian equations of motion (3.16)-(3.20). Since the full analysis is tedious, we simply present some essential points and the results for the set of four first order scalar equations which we actually use. In appendix C, we will present explicit fake Killing spinors which will serve as a check that the full set of commutator conditions is satisfied.

The dilatino equation $e^h \gamma^r \hat{\mathcal{D}} \epsilon = 0$ reads

$$(\phi' - 6\, e^h W' \gamma^r + 2iY a' e^{-A} \gamma^t)\epsilon = 0\,. \tag{4.8}$$

It is essential to use this constraint on ϵ to derive the integrability conditions. For example, if we multiply (4.8) by $(\phi' + 6\, e^h W' \gamma^r - 2iY a' e^{-A} \gamma^t)$, we obtain the scalar equation

$$\phi'^2 = 36 W'^2 e^{2h} + 4Y^2 a'^2 e^{-2A}\,. \tag{4.9}$$

From commutators not involving $\mathcal{D}_r$ we obtain the three additional equations

$$B'W' = -\frac{1}{3} W\phi' \tag{4.10}$$

$$A'\phi' = -12\, e^{2h} W W' - 16 XY a'^2\, e^{-2A} \tag{4.11}$$

$$A'B' + B'^2 = 8 e^{2h} W^2 - 16 X^2 a'^2 e^{-2A} + e^{2h-2B}. \tag{4.12}$$

This system of first order coupled differential equations can be solved exactly, as we demonstrate next. The specific functions $X(\phi)$, $Y(\phi)$, $Q(\phi)$, $W(\phi)$ given in (3.6) are used to obtain the solution.

4.2 Construction

We start by integrating the gauge equation of motion (3.19), and write its square as

$$a'^2\, e^{-2A} X^2 = \sigma^2 X^{-2} e^{2h-6B}. \tag{4.13}$$

where σ is an integration constant. This relation may be inserted in (4.9) and the four conditions (4.9)-(4.12) then reduce to coupled equations for the metric functions A, B, h. To solve them it is useful to treat A, B, h as functions of the scalar ϕ. Temporarily introducing a dot to denote derivatives with respect to ϕ, the equations become

$$e^{-2h}\phi'^2 = 36\dot{W}^2 + 144 k^2 \sigma^2 X^{-2} e^{-6B}\,, \tag{4.14}$$

$$\dot{B} = -\frac{1}{3}\frac{W}{\dot{W}}\,, \tag{4.15}$$

$$e^{-2h}\phi'^2 \dot{A} = -12 W\dot{W} - 96 k\sigma^2 X^{-2} e^{-6B}\,, \tag{4.16}$$

$$e^{-2h}\phi'^2 (\dot{A}\,\dot{B} + \dot{B}^2) = 8W^2 - 16\sigma^2 X^{-2} e^{-6B} + e^{2B}\,. \tag{4.17}$$

Plugging equations (4.14)-(4.16) into the l.h.s. of eq. (4.17) we find a very simple algebraic equation for B,

$$e^{-2B} = \left| \frac{1}{4\sigma} X \frac{\dot{W}}{\dot{W} + kW} \right|\,. \tag{4.18}$$

However, eq. (4.15) can also be integrated directly using the superpotential in (3.6). Including a constant of integration, c_B, we find

$$e^{-2B} = c_B \left(e^{k\phi} - e^{-\frac{2}{3k}\phi} \right) = \frac{c_B\,(2+3k^2)L}{k}\, e^{k\phi - \frac{2}{3k}\phi}\, \dot{W}(\phi)\,. \tag{4.19}$$

The expressions for $e^{-2B(\phi)}$ in (4.18) and (4.19) are proportional. Requiring equality fixes the relationship between the two integration constants σ and c_B,

$$|\sigma c_B| = \frac{1}{8\sqrt{3(2+3k^2)^3}} \,. \tag{4.20}$$

This can be understood as a condition that the first order equations (4.14)-(4.17) are mutually consistent.

Next integrate (4.16) to find

$$e^{2A} = c_A\, e^{-\frac{4}{3k}\phi}\left[c_B L^2 + e^{\frac{2}{k}\phi}\left(e^{\frac{2+3k^2}{3k}\phi} - 1\right)^{-1}\right] . \tag{4.21}$$

It remains to find the scalar profile $\phi(r)$. Although (4.14) gives a separable equation, it is difficult to integrate, so we proceed differently. We note that static 5D black holes in the literature (see [16] and references therein) are most simply described via the point singular harmonic function $H(y) = 1 + q/y^2$, and that H and ϕ are related logarithmically. We introduce H in two stages, first defining

$$\phi(H) \equiv \frac{3k}{2+3k^2}\log H\,, \tag{4.22}$$

where the multiplicative constant was chosen to simplify (4.19) and (4.21).

We temporararily regard H as the radial coordinate, $r = H$, which means that $\phi' = 3k/[(2+3k^2)H]$. Then (4.14) immediately determines h as a function of H:

$$\begin{aligned} e^{-2h} &= 4(2+3k^2)^2\, H^2\left(\dot{W}^2 + 4k^2\sigma^2 X^{-2} e^{-6B}\right) \\ &= \frac{4}{L^2 H^p}(H-1)^2\Big[H^{3p} + c_B L^2(H-1)\Big], \end{aligned} \tag{4.23}$$

in which we have introduced the constant

$$p = \frac{2}{2+3k^2}\,. \tag{4.24}$$

The scale factors of the metric can then be written as

$$e^{2A} = c_A L^2 H^{-2p}\left(c_B + \frac{1}{L^2(H-1)}H^{3p}\right), \tag{4.25}$$

$$e^{2B} = \frac{1}{c_B(H-1)}H^p\,, \tag{4.26}$$

$$e^{2h} = \frac{1}{4(H-1)^3}H^p\left(c_B + \frac{1}{L^2(H-1)}H^{3p}\right)^{-1} . \tag{4.27}$$

The line element now contains the term $e^{2h}dH^2$.

The scale factors e^{2A} and e^{2B} diverge as H approaces $H = 1$, indicating that this is asymptotic infinity (where we will find an AdS_5 boundary). The scalar $\phi \to 0$ in this limit, and $\phi = 0$ is the unique root of $W'(\phi) = 0$ and is a local maximum of the potential $V(\phi)$. We therefore introduce the radial coordinate y such that $H \to 1$ for $y \to \infty$, i.e.

$$H(y) = 1 + \frac{q}{y^2}\,. \tag{4.28}$$

Here q is a constant of dimension $(\text{length})^2$.

The radial term in the line element now becomes

$$e^{2h}\,dH^2 = e^{2h}(\partial_y H)^2\,dy^2 = H^p\left(c_B\,q + \frac{y^2}{L^2}H^{3p}\right)^{-1} dy^2 \tag{4.29}$$

and, the other scale factors are

$$e^{2A} = \frac{c_A L^2}{q} H^{-2p}\left(c_B\,q + \frac{y^2}{L^2}H^{3p}\right), \tag{4.30}$$

$$e^{2B} = \frac{1}{q\,c_B}\,y^2 H^p\,. \tag{4.31}$$

The metric of pure AdS_5 can be written as

$$ds^2 = -\left(1+\frac{y^2}{L^2}\right)dt^2 + \left(1+\frac{y^2}{L^2}\right)^{-1} dy^2 + y^2\,d\Omega_3\,. \tag{4.32}$$

We require that our metric match the leading terms of (4.32) as $y \to \infty$, and this fixes the remaining integration constants to be

$$c_A = \frac{q}{L^2}\,, \qquad c_B = \frac{1}{q}\,. \tag{4.33}$$

Thus we can write the general solution to the first order equations (4.9)-(4.12)

$$ds^2 = -H^{-2p} f\,dt^2 + H^p f^{-1}\,dy^2 + y^2 H^p\,d\Omega_3\,, \tag{4.34}$$

where

$$H(y) = 1 + \frac{q}{y^2}\,, \qquad f(y) = 1 + \frac{y^2}{L^2}\,H^{3p}\,. \tag{4.35}$$

(The metric is not conformally flat.)

The scalar is

$$e^{\phi} = H^{\frac{3kp}{2}}\,, \tag{4.36}$$

and the gauge field strength is found from (4.13)

$$F_{yt} = a' = \sigma X^{-2} e^{A-3B+h} = 48(2+3k^2)\,\sigma\,y^{-3}H^{-2}\,. \tag{4.37}$$

Using (4.20) and (4.33) we integrate (4.37) to find

$$A_t = a = \mp\sqrt{\frac{3}{2+3k^2}}\,\frac{q}{q+y^2} = \pm\sqrt{\frac{3}{2+3k^2}}\,(H^{-1}-1)\,, \tag{4.38}$$

where we have fixed the constant of integration such that $A_t \to 0$ for $y \to \infty$.[4]

We have constructed the most general solutions of the first order equations derived from integrability of the fake Killing spinor equations. We call them fake BPS solutions.

[4] The sign of the gauge potential and electric field are arbitrary and independent of the sign of q.

For each value of the parameters k and L from the fake supergravity action, there is a 1-parameter set of solutions depending on q. The solutions carry electric charge which can be calculated from the integral

$$q_{\rm elec} = \frac{1}{2\pi^2} \int_{S^3} Q \star F \tag{4.39}$$

over the asymptotic 3-sphere. The result is

$$q_{\rm elec} = \pm 2 \sqrt{\frac{3}{2+3k^2}}\, q\,. \tag{4.40}$$

Since there are no charged sources for the gauge field, this charge is concentrated at the center of the S^3, where the scale factor $e^{2B} = y^2 H^p$ vanishes.

4.3 Non-extremal solutions

The solutions constructed above can be generalized beyond extremality. Inspired by the solutions of [13] we simply modify f and A_t to be

$$f(y) = 1 + \frac{y^2}{L^2} H^{3p} - \frac{\mu}{y^2} \tag{4.41}$$

and

$$A_t = -\frac{\tilde{q}}{q} \sqrt{\frac{3}{(2+3k^2)}} \left(H^{-1} - 1\right), \tag{4.42}$$

where

$$\tilde{q}^2 = q(q+\mu)\,. \tag{4.43}$$

It is straightforward to verify that the equations of motion (3.16)-(3.20) are satisfied for all k, but the first order BPS equations are no longer satisfied. The electric charge is changed to

$$q_{\rm elec} = \pm 2 \sqrt{\frac{3}{2+3k^2}}\, \tilde{q}\,. \tag{4.44}$$

The mass of the solutions with respect to the background AdS_5 space is

$$M_0 = \frac{\pi}{4G} \left[\frac{3\mu}{2} + \frac{6}{2+3k^2}\, q \right]. \tag{4.45}$$

This is computed in section 5, where we also discuss a BPS bound.

4.4 Comparison with known solutions

It turns out that none of the extremal solutions found above by the fake supergravity technique, and none of their non-extremal extensions, are new. We review relevant past work here beginning with solutions found in the AdS/CFT context.

Superpotentials which are the sum of two exponentials, as in (3.6), have occurred before in applications of 5D supergravity, namely in [24, 25]. There flat sliced domain wall solutions with no gauge fields were found. These Coulomb branch solutions lift to type IIB supergravity and correspond to continuous distributions of D3-branes on subspheres of the S^5 of dimension $n = 1, \ldots, 5$. In fact from eq. (15) of [24] (after the change $\mu = \pm\phi/\sqrt{2}$ to agree with our conventions), one can see that the five superpotentials considered there agree with our $W(\phi)$ for the specific values of k

$$\begin{array}{rccccc} k = & \sqrt{\frac{10}{3}}, & \frac{2}{\sqrt{3}}, & \sqrt{\frac{2}{3}}, & \frac{1}{\sqrt{3}}, & \sqrt{\frac{2}{15}}, \\ n = & 1, & 2, & 3, & 4, & 5\,. \end{array} \tag{4.46}$$

There is an even closer relation to our work. Namely, the scale factor A of the flat-sliced ansatz (2.20), expressed as a function of the scalar field as $A(\phi)$ obeys the same equation (4.15) as our $B(\phi)$ and has the same solution.

For $k = 2/\sqrt{3}$ and $1/\sqrt{3}$ the theory (3.1) can be recognized as special cases of the supersymmetric $\mathrm{U}(1)^3$ theory which again is a consistent truncation of Type IIB supergravity on S^5. The $\mathrm{U}(1)^3$ theory consistently truncated to a single scalar field includes two gauge fields [16]. The two solutions with $k = 2/\sqrt{3}$ and $1/\sqrt{3}$ correspond to setting either of those gauge fields to zero.[5] Solutions of the further truncated theory, for which the scalars completely decouple, leaving 5D minimal gauged supergravity, can be obtained as the $k \to 0$ limit (appropriately defined) of our solutions. Since they can be embedded as solutions of the supersymmetric $\mathrm{U}(1)^3$ theory, the fake BPS solutions are in the three cases, $k = 0,\ 1/\sqrt{3},\ 2/\sqrt{3}$, truly supersymmetric [11, 12, 14]. Their non-extremal generalizations coincide with those of [13]. The 10D lifts [15, 18] of these solutions, known as "superstars", describe distributions of giant gravitons rotating on the S^5 [14].

One might hope to lift the other three values of k from table (4.46). This requires going beyond the $\mathrm{U}(1)^3$ truncation, for example to the gauged SO(6) truncation [26]. However, it appears that the gauge kinetic function of [26] is not compatible with our $Q(\phi)$ for the relevant values of k [27]. It seems unlikely that the solutions can be lifted to 10D for general k.

The extremal and non-extremal solutions for general k are not new, but were found in [19].[6] Ref. [19] constructed similar solutions for any $D \geq 4$. For general $D \geq 4$ the scalar potential of [19] can indeed be constructed from our superpotential (A.11).

4.5 Scalar "flow" and horizons

The scalar potential (1.3) constructed from the superpotential (3.6) is

$$V(\phi) = -\frac{6}{(2+3k^2)^2\,L^2}\left[18k^2\, e^{\left(\frac{2}{3k}-k\right)\phi} + 3k^2(3k^2-1)\, e^{\frac{4}{3k}\phi} - (3k^2-4)\, e^{-2k\,\phi}\right]. \tag{4.47}$$

The behavior of V depends on the value[7] of $0 < k < \infty$, but in all cases there is a local maximum at $\phi = 0$, which occurs at the AdS boundary of the solutions.

[5] This is not a consistent truncation for $k = 1/\sqrt{3}$, because the gauge field set to zero is then sourced by a potentially non-vanishing $F \wedge F$ term. Note that in our analysis of linear supergravity it was in fact only the $k = 0$ and $k = 2/\sqrt{3}$ cases which were fully linearly supersymmetric (see appendix A for details).

[6] The solutions of [19] are presented using a different coordinate system; it is easy to relate their choice of radial coordinate to our y.

[7] We restrict to $k > 0$, since $k \to -k$ is equivalent to taking $\phi \to -\phi$.

JHEP12(2007)023

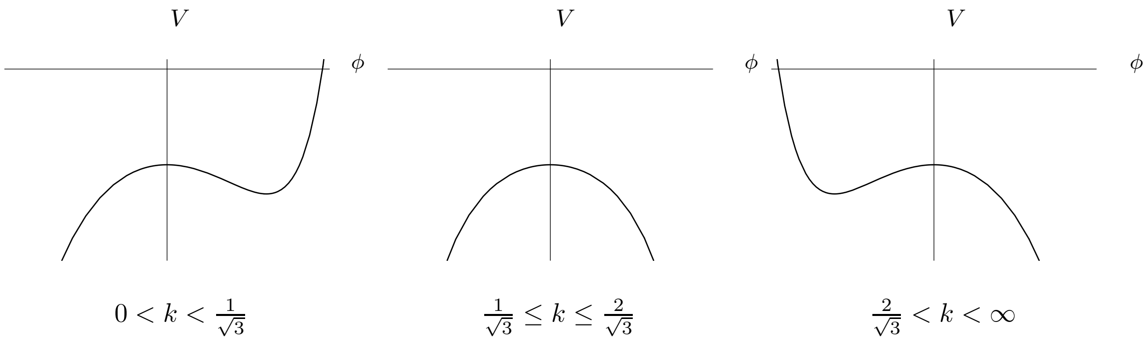

Figure 1: The behavior of the potential V for various values of k. For all k the potential has a local maximum at $\phi = 0$ so that $V(0) = -6/L^2$. Our solutions represent a "flow" from AdS at the top of the local maximum at $\phi = 0$ towards $\phi \to \infty$ when $q > 0$, and towards $\phi \to -\infty$ when $q < 0$.

When $1/\sqrt{3} \leq k \leq 2/\sqrt{3}$, the maximum at $\phi = 0$ is global, but for $0 < k < 1/\sqrt{3}$ or $k > 2/\sqrt{3}$, the potential has a local minimum located at

$$\phi_{\min} = \frac{3k}{2+3k^2} \log\left(\frac{3k^2-4}{2(3k^2-1)}\right) . \tag{4.48}$$

Note that $\phi_{\min} < 0$ for $k > 2/\sqrt{3}$, while $\phi_{\min} > 0$ for $0 < k < 1/\sqrt{3}$. The behavior of the potential is sketched in figure 1. The local minimum appears to be of little significance for the solutions, since the scalar is not stationary there due to the presence of a non-vanishing electric field.

Solutions with $q > 0$

When $q > 0$, the range of the coordinate y is 0 to +infinity: $y \to +\infty$ is the asymptotic AdS region, and $y = 0$ is the location of a curvature singularity. Since $H(y)$ in (4.28) is positive, the scalar ϕ (4.22) is non-negative. It flows from $\phi = 0$ at the boundary to $\phi \to +\infty$ at the singularity.

It is possible to hide the curvature singularity behind an event horizon for $q > 0$ by turning on the non-extremality parameter μ. The horizon is located at the (largest) zero of the function f in (4.41). The conditions for the existence of a horizon are analyzed in appendix B and we summarize the result here:

- For $k > 1/\sqrt{3}$ the solution has a single horizon whenever $\mu > 0$.
- For $k = 1/\sqrt{3}$ the existence of a horizon requires $\mu > q^2/L^2$. There is no inner horizon.
- For $k < 1/\sqrt{3}$ an event horizon requires $\mu \geq \mu_k(q, L)$, where $\mu_k(q, L)$ solves (B.6)-(B.7), as described in appendix B. Whenever $\mu > \mu_k(q, L)$ the solution has an inner horizon in addition to the event horizon. For $\mu = \mu_k(q, L)$ the horizons coincide, and the solutions are extremal but not fake BPS.

Consider a solution with a horizon located at $y = y_h$. The Hawking temperature is

$$T_{\mathrm{H}} = \frac{1}{4\pi} \frac{f'(y_h)}{[H(y_h)]^{3p/2}} , \tag{4.49}$$

and the entropy S, computed from the horizon area $A_{\rm H}$, is

$$S = \frac{A_{\rm H}}{4G} = \frac{\pi^2}{2G}\,[H(y_h)]^{3p/2}\,y_h^3\,. \tag{4.50}$$

We note in particular that the extremal non-fake-BPS solutions which exist for $k < 1/\sqrt{3}$ are characterized by having $f(y_h) = f'(y_h) = 0$. Hence these have zero temperature and finite horizon area.

The superstar cases, $k = 1/\sqrt{3}$ and $2/\sqrt{3}$, are the borderline cases for the behavior of the potential in figure 1. For $k = 2/\sqrt{3}$ the non-extremal superstars have horizons when $\mu > 0$. For $k = 1/\sqrt{3}$ a horizon exists when $\mu > q^2/L^2$.

Solutions with $q < 0$

When $q < 0$, the y-coordinate ranges from the boundary $y \to \infty$ to $y = \sqrt{|q|}$, where there is a naked singularity. Note that $0 < H(y) < 1$, so ϕ is negative. Referring to figure 1, the scalar "flows" from AdS at the top of the local maximum towards the singularity at $\phi \to -\infty$.

It is *not* possible to hide the singularity behind a horizon for any value of μ when $q < 0$. Note that the non-extremality parameter μ affects the electric field since $\tilde{q} = \sqrt{q(q+\mu)}$. A real electric field requires that

$$\tilde{q}^2 = |q|(|q| - \mu) > 0, \qquad \text{i.e.} \qquad \mu < |q| \tag{4.51}$$

However, with (4.51), we see from (4.41) that $f(y) > 0$ for $y \geq \sqrt{|q|}$. Thus we conclude that for a physical electric field, one cannot have a non-extremal solution in which the naked singularity is shielded.

It was proposed in [28] that a spacetime with a naked singularity may be considered physical if the solution generalizes to one in which the singularity is hidden behind an event horizon. This is not satisfied by the $q < 0$ solutions, which also fail another criterium [29], namely that g_{tt} should not diverge at the singularity, since that violates the UV/IR connection. Moreover, we show in section 5 that the mass of the fake BPS solutions with $q < 0$ is negative. Each of these observations indicates that the solutions with $q < 0$ are unphysical.

5. Mass and charge from holography

In this section we derive properties of the boundary field theory from the AdS/CFT correspondence. We will use the formalism of holographic renormalization in which field theory observables are calculated from a properly renormalized on-shell action involving the boundary limit of the bulk fields of our system. This formalism was developed in several papers; for example see [23, 32–35].

5.1 Holographic stress-energy tensor

The form of the boundary action depends on the bulk fields and their mutual interaction. Fortunately the relevant holographic observables were derived in [36] for the same bulk system we are studying, namely the metric, a scalar dual to a $\Delta = 2$ operator, and massless gauge fields with kinetic term modified by an exponential function of the scalar. In fact for the specific values $k = 1/\sqrt{3}$ and $2/\sqrt{3}$, the bosonic Lagrangian (1.5), with (3.6), agrees in full detail with the system studied in [36]. The scalar sector of the matter system is invariant under the change $k \to 2/(3k)$, but the gauge field sector differs for these two values. Gauge field fluctuations were added to the system in [37] and further studied in [36]. There is an $\mathrm{SO}(4) \times \mathrm{SO}(2)$ flavor symmetry, and it turns out that the SO(2) gauge field couples as in our system for the case $k = 2/\sqrt{3}$, while SO(4) gauge fields correspond to $k = 1/\sqrt{3}$.

In the holographic renormalization formalism of [36] the bulk metric is parameterized by

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(\rho, x^i)\, dx^i dx^j\,, \tag{5.1}$$

in which ρ is the radial variable, and the x^i are an arbitrary set of coordinates of the boundary at $\rho = 0$. Thus our first task is to relate ρ to the radial variable y of (4.34). We need this relation near the boundary, so we solve the equation

$$\frac{dy}{d\rho} = -\frac{L}{2\rho} f^{1/2} H^{-p/2}\,, \tag{5.2}$$

as the series

$$y \sim \frac{L}{\sqrt{\rho}}(1 + a_1\rho + a_2\rho^2 + \dots)\,, \tag{5.3}$$

where the coefficients are given by

$$a_1 = -\frac{1}{4} - \frac{q}{(2+3k^2)L^2}\,, \tag{5.4}$$

$$a_2 = \frac{\mu}{8L^2} + \frac{q}{4(2+3k^2)L^2} - \frac{q^2(2-3k^2)}{4(2+3k^2)^2 L^4}\,. \tag{5.5}$$

After reexpression in terms of ρ, the bulk fields $g_{ij}(\rho, x^i)$, $A_\mu(\rho, x^i)$, $\phi(\rho, x^i)$ of our solution have expansions in the coordinate ρ which are determined by the boundary limit of the field equations. In general, both powers of ρ and $\ln(\rho)$ occur in these expansions, but it is obvious from the solution (1.7) that there are no logarithms in our case. We omit them in the expansions which then take the simpler form:

$$g_{ij} = g^{(0)}_{ij} + g^{(2)}_{ij}\rho + g^{(4)}_{ij}\rho^2 + \cdots \tag{5.6}$$

$$\phi = \phi^0\rho + \phi^{(2)}\rho^2 + \cdots \tag{5.7}$$

$$A_i = A^{(0)}_i + A^{(2)}_i\rho + \cdots \tag{5.8}$$

The leading term $g^{(0)}_{ij}$ of the expansion (5.6) is the spacetime metric for the boundary gauge theory. In our case this is the metric

$$ds_4^2 = g^{(0)}_{ij} dx^i dx^j = -dt^2 + L^2 d\Omega_3\,. \tag{5.9}$$

JHEP12(2007)023

We have chosen the constant of integration in (5.3), so that L is the radius of the boundary S^3. Note that the scalar "source rate" term, which would be proportional to $\rho \ln \rho$ is absent for us and the leading term of ϕ vanishes at the "vev rate". For the gauge field, which is in the gauge $A_\rho = 0$, the expansion of A_i is more general than actually occurs. Namely, only the component A_t is non-vanishing, and its source rate term $A_t^{(0)}$ vanishes, leaving the vev rate $A_t^{(2)}\rho$ as the leading term.

We now apply the relevant formulas of sections 5–6 of [36] after conversion to our conventions.[8] Formula (5.45) of [36] gives the 1-point function of the field theory stress tensor:

$$\begin{aligned}\langle T_{ij}\rangle = \frac{1}{4\pi G\, L}\Big\{ & g^{(4)}_{ij} + \frac{1}{8}\Big[\mathrm{Tr}\,(g^{(2)})^2 - (\mathrm{Tr}\,g^{(2)})^2\Big] g^{(0)}_{ij} - \frac{1}{2}(g^{(2)})^2_{ij} \\ & + \frac{1}{4} g^{(2)}_{ij}\,\mathrm{Tr}\,g^{(2)} + \frac{1}{6}(\phi^{(0)})^2 g^{(0)}_{ij} + \frac{3}{2} h^{(4)}_{ij}\Big\}\,, \end{aligned} \tag{5.10}$$

in which all contractions are taken with $g^{(0)}_{ij}$. The expression for $h^{(4)}_{ij}$ is given in (5.38) of [36]. It involves various contractions of the curvature tensor of the boundary metric and vanishes for the metric (5.9). The effect of the background gauge field was not considered in section 5 of [36], but it can be seen to vanish at the rate ρ^2 at the boundary and thus not contribute to the 1-point function (5.10). Using the quite general holographic formula $g^{(2)}_{ij} = -L^2(R_{ij} - g^{(0)}_{ij}\, R/6)/2$, one can show that the two terms in $\big[\ldots\big]$ cancel, so that (5.10) reduces to

$$\langle T_{ij}\rangle = \frac{1}{4\pi G\, L}\Big[g^{(4)}_{ij} - \frac{1}{2}(g^{(2)})^2_{ij} + \frac{1}{4} g^{(2)}_{ij}\,\mathrm{Tr}\,g^{(2)} + \frac{1}{6}(\phi^{(0)})^2 g^{(0)}_{ij}\Big]\,. \tag{5.11}$$

We can now evaluate this 1-point function by applying the coordinate relation (5.3) to the various contributions to the bulk solution. We then obtain the stress tensor

$$\langle T_{tt}\rangle = \frac{1}{4\pi G\, L}\left[\frac{3}{16} + \frac{3\mu}{4L^2} + \frac{3}{(2+3k^2)L^2}\, q\right], \tag{5.12}$$

$$\langle T_{\theta\theta}\rangle = \frac{1}{4\pi G\, L}\left[\frac{L^2}{16} + \frac{\mu}{4} + \frac{1}{(2+3k^2)}\, q\right], \tag{5.13}$$

for the field theory dual of the non-extremal solutions. The mass, $M = \int_{S^3}\langle T_{tt}\rangle = 2\pi^2 L^3\,\langle T_{tt}\rangle$, is then

$$M = \frac{\pi}{4G}\left[\frac{3L^2}{8} + \frac{3\mu}{2} + \frac{6}{2+3k^2}\, q\right]\,. \tag{5.14}$$

The first term proportional to L^2 is the Casimir energy [32]. Substracting it we have

$$M_0 = \frac{\pi}{4G}\left[\frac{3\mu}{2} + \frac{6}{2+3k^2}\, q\right]\,. \tag{5.15}$$

For extremal solutions one sets $\mu = 0$. Note that using our asymptotic charge q_{elec} in (4.40), the mass formula suggests a fake BPS bound

$$M_0 \geq \frac{\pi}{4G}\sqrt{\frac{3}{2+3k^2}}\; q_{\mathrm{elec}}\,. \tag{5.16}$$

[8]The scalar ϕ of [36] is multiplied by $1/\sqrt{2}$, and we reinstate dimensions by including a factor of $\frac{1}{4\pi G\, L}$.

We will examine this using the Witten-Nester method in section 5.2.

Note that the trace of the stress-energy tensor vanishes exactly,

$$\langle T^i{}_i \rangle = -\langle T_{tt} \rangle + \frac{3}{L^2} \langle T_{\theta\theta} \rangle = 0 \,. \tag{5.17}$$

This must be the case because the holograhic trace anomaly [23] is proportional to $R^{\mu\nu} R_{\mu\nu} - R^2/3$ and this vanishes for the $\mathbb{R} \times S^3$ boundary metric.

Holographic renormalization also determines precise formulas for the vevs of the operator $\mathcal{O}_\phi$ dual to the bulk scalar and the conserved current J^t dual to A_t. For the cases $k = 1/\sqrt{3}, 2/\sqrt{3}$, for which our solutions agree with the superstars, $\mathcal{O}_\phi$ is a component of $\mathrm{Tr}(X^2)$ and J^t is the time component of a conserved R-current of $\mathcal{N} = 4$ SYM theory. Formula (5.33) of [36] gives the scalar vev

$$\langle \mathcal{O}_\phi \rangle = \frac{1}{L^2} \sqrt{2} \, \phi^{(0)} = \frac{3\sqrt{2} \, k \, q}{(2 + 3k^2) L^4} \,, \tag{5.18}$$

where $\phi^{(0)}$ is given by (5.7). From formula (6.77) of [36] we find that

$$\langle J^t \rangle = \frac{2}{L^3} \, A_t^{(2)} = 2 \sqrt{\frac{3}{2 + 3k^2}} \, \frac{\tilde{q}}{L^5} \,. \tag{5.19}$$

Note that

$$\frac{1}{2\pi^2} \int_{S^3} \langle J^t \rangle = \frac{q_{\mathrm{elec}}}{L^2} \,, \tag{5.20}$$

with q_{elec} given by (4.44).

5.2 Witten-Nester in fake supergravity

In this section we use the Witten-Nester method to calculate the energy of our solutions. This method determines the energy of a spacetime with respect to a background spacetime — such as flat Minkowski or anti-de Sitter space.

The Witten-Nester energy E_{WN} is defined as the asymptotic boundary integral

$$E_{\mathrm{WN}} = \int_{\partial\Sigma} \star \hat{E} \,, \qquad i \hat{E}^{\mu\nu} = \bar{\epsilon}_1 \Gamma^{\mu\nu\rho} \mathcal{D}_\rho \epsilon_2 - \bar{\epsilon}_1 \overleftarrow{\mathcal{D}}_\rho \, \Gamma^{\mu\nu\rho} \epsilon_2 \,. \tag{5.21}$$

$\hat{E}$ is the Witten-Nester 2-form and $\mathcal{D}_\rho \epsilon$ is given by the fake gravitino transformation rule in (4.1). The spinors ϵ must asymptotically approach Killing spinors of the reference background, in this case AdS_5. Via Stokes' theorem, the boundary integral (5.21) can be converted to the bulk integral

$$E_{\mathrm{WN}} = \int_{\Sigma} d(\star \hat{E}) \,. \tag{5.22}$$

This results holds *only if* there are no contributions from naked singularities or horizon boundaries. In our applications below we assume that μ is large enough that our solutions have regular horizons (cf. discussion in section 4.5 and appendix B). We further impose a condition on the Witten spinor ϵ so that the contribution from the horizon boundary vanishes. For details of this procedure, and discussion of the existence of Witten spinors, we refer to [38].

The standard approach of Witten-Nester is to show that the bulk integral (5.22) is positive (semi)definite and vanishes only for solutions which admit Killing spinors. The boundary integral contains the conserved charges (mass, electric charge, angular momentum etc). Combining the information from the bulk and boundary integrals a BPS bound, or positive energy statement, is derived.

A direct calculation of the bulk integral (5.22) with $\epsilon_1 = \epsilon_2$ gives

$$\text{Bulk}: \qquad E_{\text{WN}} = -i \int_\Sigma d\Sigma_\mu \Big[2\, \overline{\delta\psi}_\nu \Gamma^{\mu\nu\rho} \delta\psi_\rho - \frac{1}{2}\, \overline{\delta\lambda}\, \Gamma^\mu\, \delta\lambda \qquad (5.23)$$
$$- \frac{1}{2}\, i\, \epsilon^{\mu\lambda\kappa\rho\sigma} F_{\lambda\kappa} F_{\rho\sigma} \Big(Y^2 - 48 X^2 + 72 b\, X Q^{-1} \Big)\, \bar\epsilon\, \epsilon \Big] \,.$$

This result depends on the form of the fake supergravity transformations, the identities (3.6) and the Einstein and gauge field equations. It is valid for any solution of the equations of motion. The first two contributions to the $F \wedge F$-term come from γ-matrix identities, while the last one comes from including a Chern-Simons term (A.14) with coefficient b.

Using $\bar\epsilon = \epsilon^\dagger i \gamma^t$ and the Witten condition $\Gamma^a \mathcal{D}_a \epsilon = 0$, the first two terms of (5.23) can be shown to be positive definite. Had it not been for the $F \wedge F$-term we would use this to derive a general BPS bound relating mass and charge. The coefficient of the $F \wedge F$-term is identical to the first condition of (A.16) which was obtained by requiring the full action to be linearly supersymmetric. As concluded in appendix A, the condition can only be satisfied for $k = \pm 2/\sqrt{3}$ and $b = 0$, or for $k = 0$ and $b = 1/(6\sqrt{6})$.[9]

For the class of solutions with $F \wedge F = 0$, including our static electrically charged generalized superstar solutions, we get a bound as follows. The boundary integral can be evaluated by treating the spacetime of interest as a fluctuation about the asymptotic background; in our case this background is global AdS space. There are contributions from the metric, the scalar and the gauge field. For our solutions the Witten-Nester boundary integral gives

$$\text{Boundary}: \qquad E_{\text{WN}} = 2\pi^2\, v_1^\dagger \left\{ \left[\frac{3\mu}{2} + \frac{6\, q}{2 + 3k^2} \right] + \frac{6\, \tilde q}{2 + 3k^2}\, i\gamma^t \right\} v_2 \,, \qquad (5.24)$$

with v_1 and v_2 denoting unconstrained constant spinors. The first term of (5.24) is the mass. The term with $i\gamma^t$ comes from the F_{yt}-term in the S^3 components of $\mathcal{D}_\mu \epsilon$ in (4.1) and is proportional to the charge. The Witten-Nester argument tells us that the hermitean symmetric quadratic form in (5.24) is non-negative, and we thus derive the inequality

$$M_0 \equiv \frac{3\mu}{2} + \frac{6\, q}{2 + 3k^2} \;\; \geq \;\; \frac{6}{2 + 3k^2} |\tilde q| = \sqrt{\frac{3}{2 + 3k^2}}\, |q_{\text{elec}}| , \qquad (5.25)$$

in which q_{elec} is the electric charge (4.44) of the non-extremal solution. This is the bound anticipated from the holographic calculation.

[9] As discussed in section 4.4, the supersymmetric theory of the other superstar case, $k = \pm 1/\sqrt{3}$, contains an extra gauge field which must be included in order to obtain a general Witten-Nester bound.

One might have expected that the fake supergravity framework would have allowed the derivation of a general bound relating energy and charge. But this is false because the $F \wedge F$-terms in (5.23) do not have the required positivity. The same conclusion holds for any dimension $D \geq 5$, where a fifth rank γ-matrix give the analogous non-positive $F_{\lambda\kappa}F_{\rho\sigma}$-terms. In $D = 4$, however, the fifth rank γ-matrix vanishes identically, so linear supergravity is complete and a general BPS bound can be derived.

6. Discussion

We have extended the method of fake supergravity with the purpose of exploring the AdS/CFT correspondence for field theories on $\mathbb{R} \times S^3$. An abelian gauge field has been included in order to obtain non-trivial solutions. The bulk gauge field restricts the bulk Lagrangian and leaves only a choice of a real constant k which appears in the exponential $e^{2k\phi}$ of the coupling between the scalar and the gauge field. The fake supersymmetric electrically charged solutions of the $D = 5$ first order flow equations, are generalizations of "superstar" solutions. For special values of k these are truly supersymmetric superstars. Non-extremal generalizations include a Schwarzschild-like parameter which makes it possible the hide the otherwise nakedly singular source of the electric field behind an event horizon.

This work was originally motivated by the wish to find holographic duals of renormalization group flows for field theories on $\mathbb{R} \times S^3$. As it turned out our solutions describe duals of states rather than deformations of $\mathcal{N} = 4$ SYM theory. It is possible that fake supergravity will lead to new flows when carried out for a bulk theory with more fields and perhaps with a solution ansatz which only has the symmetry $\mathbb{R} \times S^3$ asymptotically.

Fake supergravity has been applied to describe holographic renormalization group flows and to the problem of (classical) stability. The applications also include a correspondence between domain walls and cosmology solutions through analytic continuations [39]. Recently, a different formulation of fake supergravity has been used to find first order flow equations for $D = 4$ non-supersymmetric black hole solutions [40]. The diversity of the applications demonstrate the power of the method.

Our work has illustrated the use of fake supergravity for finding and solving first order flow equations, even in cases where the action is linearly supersymmetric only for a certain class of field configurations which include the solution ansatz. This in turn revealed a limitation in the application of fake supersymmetry to derivations of BPS bounds. The result indicates a connection between obtaining general linear fake supersymmetry of the action and achieving a BPS-type bound on the Witten-Nester energy. It would be interesting to establish such a connection in more general contexts.

Acknowledgments

We would like to thank Mirjam Cvetic, Michael Duff, Gary Gibbons, Matt Headrick, Micheal Kiermaier, Albion Lawrence, Hong Lü, Oleg Lunin, John McGreevy, Bob McNees, Ioannis Papadimitriou, Chris Pope, and Amit Sever for useful discussions. HE is supported by a Pappalardo Fellowship in Physics at MIT. DZF is supported by NSF grant PHY-0600465. HL is supported in part by the Alfred P. Sloan Foundation and U.S. Department of Energy (D.O.E) OJI grant. In addition, all three authors are supported by the US Department of Energy through cooperative research agreement DE-FG0205ER41360.

A. Details of the fake supergravity construction

This appendix provides more information on the determination of the various scalar functions in the fermion action and transformation rules given in section 3. Several γ-matrix identities are used in the calculations, such as

$$\gamma^{\mu\nu\rho}\big(\gamma_\mu{}^{\sigma\tau} - 2(d-2)\delta_\mu^{\ \sigma}\gamma^\tau\big)F_{\sigma\tau} = -(d-1)\big(\gamma^{\nu\rho\sigma\tau}F_{\sigma\tau} + 2F^{\nu\rho}\big)\,, \tag{A.1}$$

in $D = d+1$ dimensions.

We begin by listing several of the $\bar\lambda(\cdots)\epsilon$ spinor bilinears which appear in $\delta(S_b + S_f + S_{\rm gauge})$ and the conditions that their vanishing imposes on the scalar functions:

$$\begin{array}{rr@{\;=\;}l} \big(\bar\lambda\gamma^\mu\gamma^{\rho\sigma}F_{\rho\sigma}D_\mu\epsilon\big) & N & Y\,,\\ \big(\bar\lambda\gamma^{\mu\rho\sigma}F_{\rho\sigma}\partial_\mu\phi\,\epsilon\big) & P & (d-1)X - Y'\,,\\ \big(\bar\lambda\gamma_\mu\partial_\nu F^{\mu\nu}\epsilon\big) & \beta\,Q & 2Y\,,\\ \big(\bar\lambda\gamma^\mu F_{\mu\nu}\partial^\nu\phi\,\epsilon\big) & \beta\,Q' & 4Y'\,. \end{array} \tag{A.2}$$

Eliminating β from the two relations in which it appears gives

$$\frac{Q'}{Q} = 2\frac{Y'}{Y}. \tag{A.3}$$

The analogues spinor bilinears involving the gravitino lead to the additional conditions

$$\begin{array}{rr@{\;=\;}l} \big(\bar\psi_\mu \overleftarrow{D}_\nu\,(\gamma^{\mu\nu\rho\sigma}F_{\rho\sigma} + 2F^{\mu\nu})\epsilon\big) & M & -2(d-1)X\,,\\ \big(\bar\psi_\mu\gamma^{\mu\nu\rho\sigma}F_{\rho\sigma}\partial_\nu\phi\,\epsilon\big) & Y+N & 4(d-1)X'\,,\\ \big(\bar\psi_\mu F^{\mu\nu}\partial_\nu\phi\,\epsilon\big) & \alpha\,Q' & -16(d-1)\,X'\,,\\ \big(\bar\psi_\mu\partial_\nu F^{\mu\nu}\epsilon\big) & \alpha\,Q & -8(d-1)\,X\,. \end{array} \tag{A.4}$$

The ratio of the two relations involving α gives

$$\frac{Q'}{Q} = 2\frac{X'}{X}. \tag{A.5}$$

Then, from (A.3) we learn that X and Y are proportional. It is then convenient to impose

$$Y = 2(d-1)k\,X\,, \tag{A.6}$$

with k a constant. From the two conditions involving N and Y above, we learn that $Y = 2(d-1)X'$. Using (A.6), we find that

$$X = c_1\,e^{k\,\phi} \tag{A.7}$$

JHEP12(2007)023

in which c_1 is an integration constant which we will fix shortly.

The vanishing condition for the coefficient of $(\bar\psi_\mu \gamma^{\mu\rho\sigma} F_{\rho\sigma}\epsilon)$ is

$$c + 4(d-1)(d-2)XW - 2(d-1)YW' = 0\,. \tag{A.8}$$

Using (A.6) for Y and the exponential solution (A.7) for X, this condition becomes a differential equation which determines the superpotential to be

$$W(\phi) = -\frac{c}{4c_1(d-1)\big((d-2)+(d-1)k^2\big)}\, e^{-k\phi} + c_2\, e^{\frac{d-2}{(d-1)k}\phi}. \tag{A.9}$$

Next we discuss the several spinor bilinears of the form $(\bar\psi\,\Gamma F^2\epsilon)$, in which Γ indicates a matrix of the Clifford algebra of 5th, 3rd, or 1st rank. The 3 types are independent, so their coefficients must vanish separately. The 5th rank case is discussed below. The 3rd rank bilinear actually vanishes due to index contractions. The 1st rank terms give us the relation,

$$Q \;=\; 4\big(4(d-2)(d-1)X^2 + Y^2\big) \;=\; 16\, c_1^2\,(d-1)\big((d-2)+(d-1)k^2\big)\, e^{2k\,\phi}, \tag{A.10}$$

after use of (A.2), (A.4), (A.6) and (A.7).

The functional form of all scalar functions in the ansatz has been determined, and we now fix the integration constants c_1, c_2 using some physical input. It is convenient to choose the constant c_1 so that $Q(\phi) \to 1$ at the boundary. We then choose c_2 so that the stationary point of $W(\phi)$ occurs at $\phi = 0$. We also normalize the superpotential so that the field equations of the theory admit AdS_D with scale L as a solution. With these conventions, the superpotential (A.9) becomes

$$W(\phi) = \frac{1}{2L\big((d-2)+(d-1)k^2\big)}\left[(d-2)e^{-k\,\phi} + (d-1)k^2 e^{\frac{d-2}{(d-1)k}\phi}\right]. \tag{A.11}$$

We regard k and L as the physical parameters of the model, and express the $\mathrm{U}(1)$ coupling c in terms of them. We can summarize these relations among the parameters as

$$c = -\frac{d-2}{2L\sqrt{\frac{d-2}{d-1}+k^2}}\,, \qquad c_1 = \frac{1}{4(d-1)\sqrt{\frac{d-2}{d-1}+k^2}}\,, \qquad c_2 = \frac{k^2}{2L\big(\frac{d-2}{d-1}+k^2\big)}\,. \tag{A.12}$$

With these choices, the potential takes the form

$$V(\phi) = -\frac{d\,(d-1)}{2L^2} - \frac{(d-2)}{L^2}\,\phi^2 + \dots \tag{A.13}$$

when $\phi \to 0$, i.e. near the AdS_D boundary. This is the potential of massive scalar in AdS_D with mass $m^2 = -2(d-2)/L^2$. The BF bound of AdS_D is $m^2_{\mathrm{BF}} = -d^2/(4L^2)$, so for $D = 5$ we saturate the bound $m^2 = m^2_{\mathrm{BF}}$, whereas for $D = 4$ or $D > 5$, the mass is strictly above the bound, $m^2 > m^2_{\mathrm{BF}}$. Thus for all $D \geq 4$, fake supergravity has lead us to theories with stable potentials V.[10]

[10] In an AdS/CFT context the bulk scalar generates a deformation of the CFT by an operator of dimension $d-2$ such as a scalar mass deformation $M^2\varphi^2$.

The vanishing conditions for 10 spinor bilinears have been used so far to determine all scalar functions of the initial ansatz. There are several other bilinears with at most rank 4 γ-matrices whose vanishing conditions can be seen to be satisfied using previous results. These do not impose new conditions.

We do discuss briefly the issue of rank 5 γ-matrices which appear in the calculation as $\Gamma^{\mu\nu\lambda\rho\sigma}F_{\mu\nu}F_{\lambda\rho}$. For $D=4$ these terms of course vanish identically and no new conditions appear. We treat the cases $D=5$ and $D>5$ separately.

For $D=5$, the 5th rank γ-matrix is proportional to the Levi-Civita symbol, $\gamma^{\mu\nu\lambda\rho\sigma} = i\epsilon^{\mu\nu\lambda\rho\sigma}$, and so these terms enter in the same form as supersymmetry variations of a Chern-Simons term. Chern-Simons terms are usually present in 5-dimensional supergravity theories. Thus we might expect it necessary to add the bosonic term

$$\mathcal{L}_{\rm CS} = b\,\epsilon^{\kappa\mu\nu\rho\sigma}A_\kappa F_{\mu\nu}F_{\rho\sigma} \tag{A.14}$$

to the Lagrangian of the fake supergravity model. Its supersymmetry variation is

$$\delta\mathcal{L}_{\rm CS} = -3i\,b\,\epsilon^{\kappa\mu\nu\rho\sigma}\Big[\alpha\left(\bar\epsilon\,\psi_\mu - \bar\psi_\mu\epsilon\right) + \beta\left(\bar\epsilon\,\Gamma_\mu\lambda + \bar\lambda\,\Gamma_\mu\epsilon\right)\Big]F_{\mu\nu}F_{\rho\sigma}\,. \tag{A.15}$$

In fact the supersymmetry variation $\delta S_{\rm F}$ contains the spinor bilinears such as $(\bar\psi_\tau\gamma^{\tau\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\epsilon)$ and $(\bar\lambda\gamma^{\mu\nu\rho\sigma}\epsilon)$ which take the same form as the two terms of (A.15) when the duality relations of the 5-dimensional Clifford algebra are used. The coefficents of each bilinear are quadratic in the scalar functions of the model. Assuming that $\mathcal{L}_{\rm CS}$ is present, the cancellation conditions of the gravitino and gaugino terms are

$$D=5: \qquad \begin{aligned} Y^2 - 48X^2 - 3b\,\alpha &= 0\,,\\ YY' - 8XY - 3b\,\beta &= 0\,. \end{aligned} \tag{A.16}$$

The second condition is the derivative of the first if and only if b is constant, ie. independent of ϕ. Of course, gauge invariance requires constant b. However, using the last relation of (A.4) together with (A.6) and (A.7), one finds that there are only two solutions to (A.16), namely (1) $k=\pm2/\sqrt{3}$ and $b=0$, and (2) $k=0$ and $b=1/(6\sqrt{6})$. Both these cases correspond to consistent truncations of the $D=5$ supersymmetric $\mathrm{U}(1)^3$ theory [16], as discussed in section 4.4.

Thus we obtain a complete fake supergravity model only for these cases. However, complete linear local supersymmetry is not really required for application to solutions with purely electric field, since the two spinor bilinears themselves vanish if F_{rt} is the only non-vanishing component of the field strength. The fake supergravity approach can succeed even if the complete linear local supersymmetry fails, provided that $\delta(S_b+S_f+S_{\rm gauge})$ vanishes for the class of solutions under study.

For $D>5$, the condition that all 5th rank Γ-matrices cancel is

$$Y^2 = 4\,d\,(d-1)X^2\,. \tag{A.17}$$

This selects the values

$$D>5: \qquad k = \pm\sqrt{\frac{d}{d-1}} \tag{A.18}$$

Again, for the purpose of fake supergravity, it is only necessary to impose the condition (A.17) if $F \wedge F$ is non-vanishing for the solutions of interest.

It is somewhat surprising that the matrix structure of the superpotential $\mathbf{W}(\phi)$, which is required for AdS_d sliced domain walls does not appear in our study. In fact the ansatz (3.3), (3.5) accommodates similar matrices at several places. For example, the spinors could be doubled and X and Y replaced by matrices. However, fake supergravity is modeled on real $D = 5$, $\mathcal{N} = 2$ supergravity in which spinor doubling occurs because real $D = 5$ supergravity requires symplectic Majorana spinors. So we consulted the form of the fermion transformation rules in [41] and found that analogues of X, Y are diagonal in the symplectic indices. Thus we assumed that X, Y are scalars, rather than matrices. Then, compatibility of (A.8) with the matrix constraint (2.21) tells us that $W(\phi)$ is also scalar.

B. Conditions for existence of a horizon

Our fake BPS solutions are all nakedly singular. The non-extremal solutions have horizons when the function f has a zero for $y > 0$. It is useful to examine the condition $f(y) = 0$ using $H = 1 + q/y^2$ as a variable instead of y, and to formulate the problem in terms of the function

$$g(H) \;=\; \frac{q^2}{L^2\, y^2} f(y)\Big|_{y\to H} \;=\; \bar{q}^2 H^{3p} + \bar{q}(H-1) - \bar{\mu}(H-1)^2\,, \tag{B.1}$$

where we have introduced dimensionless parameters $\bar{q} = qL^{-2}$ and $\bar{\mu} = \mu L^{-2}$. We focus on the case of $\bar{q} > 0$ and $\bar{\mu} > 0$. The condition for having a horizon is then that there exists an $H_0 > 1$ such that $g(H_0) = 0$. Note that $g(1) = \bar{q}^2 > 0$. Depending on the behavior of g for $H \to \infty$ there three cases:

1. Case $k > \frac{1}{\sqrt{3}}$ (i.e. $0 < p < 2/3$): For $H \gg 1$, we have $g(H) \sim -\bar{\mu}H^2 < 0$. In this case g always has a single zero for $H > 1$, and so for any values of $q > 0$ and $\mu > 0$, the solution has a horizon.

2. Case $k = \frac{1}{\sqrt{3}}$ (i.e. $p = 2/3$): For $H \gg 1$, we have $g(H) \sim (\bar{q}^2 - \bar{\mu})H^2 + (\bar{q} + 2\bar{\mu})H + \ldots$. So if $\bar{\mu} > \bar{q}^2$, g has a zero. When $\bar{\mu} \le \bar{q}^2$, it is straightforward to see that g has no local extrema for $H > 1$, and therefore $g > 0$ for $H > 1$. In conclusion, for $k = \frac{1}{\sqrt{3}}$ the solutions have horizons only if $\bar{\mu} > \bar{q}^2$, i.e. if $\mu > q^2/L^2$.

3. Case $0 < k < \frac{1}{\sqrt{3}}$ (i.e. $2/3 < p < 1$): Since $g(H) \to \bar{q}^2 H^{3p} > 0$ for $H \gg 1$, the existence of a zero of g requires that g has a local minimum $H_{\min} > 1$ such that $g(H_{\min}) \le 0$. Note that $g'(1) = \bar{q}(1 + 3p\bar{q}) > 0$, so g can only have a local minimum if it also has a local maximum. So we need g' to have two separate zeroes. That in turn requires that g'' has a zero for $H > 1$. Solving $g'' = 0$ gives

$$H^{3p-2} = \frac{2\bar{\mu}}{3p(3p-1)\bar{q}^2}\,. \tag{B.2}$$

Requiring $H^{3p-2} > 1$ in (B.2) gives

$$\bar{\mu} > \frac{3}{2}p(3p-1)\bar{q}^2 \tag{B.3}$$

as a necessary condition for having a horizon. Note that (B.3) implies $\bar{\mu} > \bar{q}^2$.

The condition is (B.3) not sufficient, so we push the analysis further to solve the "extremal" case where g has a minimum at $g = 0$; i.e. we solve the system $g(H) = 0$ and $g'(H) = 0$.

Zeroes of g' are solutions $H > 1$ to the equation

$$H^{3p-1} = \frac{2(H-1)\bar{\mu} - \bar{q}}{3p\bar{q}^2}\,. \tag{B.4}$$

Plugging this into g gives

$$3p\, g(H) = -(3p-2)\bar{\mu}H^2 + (3p-1)(\bar{q} + 2\bar{\mu})H - 3p(\bar{q} + \bar{\mu})\,. \tag{B.5}$$

Let the two zeroes of (B.5) be $H_0^{\pm}$, with $H_0^- < H_0^+$. One finds that $H_0^{\pm}$ are real and satisfy $H_0^{\pm} > 1$. However, for $H = H_0^-$, the r.h.s. of (B.4) is negative, so we discard this as a solution. Setting $H = H_0^+$ in (B.4) gives an equation that can be used to determine $\bar{\mu}$ numerically for given $\bar{q}$ and p:

$$(H_0^+)^{3p-1} = \frac{2(H_0^+ - 1)\bar{\mu} - \bar{q}}{3p\bar{q}^2}\,, \tag{B.6}$$

with

$$H_0^+ = \frac{(3p-1)(\bar{q} + 2\bar{\mu}) + \sqrt{(3p-1)^2\bar{q}^2 + 4\bar{\mu}(\bar{q} + \bar{\mu})}}{2(3p-2)\bar{\mu}}\,. \tag{B.7}$$

It can be shown that whenever the condition (B.3) holds, the r.h.s. of (B.6) is greater than 1.

To summarize, for given $\bar{q}$ and $0 < k < \frac{1}{\sqrt{3}}$, μ needs to be sufficiently large in order for a horizon to exist. Condition (B.3) is a necessary, but not sufficient, condition. Solving (B.6) and (B.7) numerically for given $\bar{q}$ and p gives the value $\bar{\mu}_k(\bar{q})$ for the "extremal" black hole solution which has minimum energy above extremality for given charge $\bar{q}$. Re-introducing the AdS scale L, we denote the above μ-bound by $\mu_k(q, L)$. When $\mu > \mu_k(q, L)$, the function g has two zeroes and the solutions have both an inner and an outer horizon.

As an example, consider $k = \sqrt{2}/3$ (i.e. $p = 3/4$) and $\bar{q} = 1$. Then condition (B.3) gives $\bar{\mu} > 45/32 \approx 1.4$, whereas solving (B.6) shows that the extremality bound is $\bar{\mu}_{k=\sqrt{2}/3}(\bar{q} = 1) \approx 2.3$ for this example.

C. Fake Killing spinors

We consider first Killing spinors for pure AdS_5 in the coordinates obtained as the $q = \mu = 0$ limit of our solutions. Their form serves as a useful starting point for the construction of fake Killing spinors for the extremal solutions above, and they play a direct role in the Witten-Nester energy computation.

In the diagonal frame for the AdS_5 metric (4.32), it is straightforward to show that the AdS_5 Killing spinor ϵ_0 can be written

$$\epsilon_0 = \Big[g_+(y)P_+ + g_-(y)P_-\Big]\, e^{-\frac{1}{2L}\gamma^t\, t}\, v\,, \tag{C.1}$$

where we have introduced the projectors $P_\pm = \frac{1}{2}(1 \pm \gamma^y)$ and functions

$$g_\pm(y) = \Big[\sqrt{f_0(y)} \mp \sqrt{f_0(y)-1}\Big]^{1/2}\,, \qquad f_0(y) = 1 + \frac{y^2}{L^2}. \tag{C.2}$$

The spinor v is one variant of the S^3 Killing spinors obtained in [30]. It satisfies

$$\bar{\nabla}_i v = -\frac{1}{2}\bar{\gamma}_i\gamma^y v\,, \tag{C.3}$$

where $\bar{\nabla}_i$ and $\bar{\gamma}_i$ denote the derivatives (including spin-connections) and γ-matrices for the unit S^3.

The two presentations of the AdS_5 metric (3.24) and (4.32) differ by the relation $y = L\sinh(r/L)$ of their radial coordinates. In the coordinate r, the Killing spinor (C.1) can be rewritten as the simple expression

$$\epsilon_0 \;=\; e^{\frac{1}{2L}\gamma^5 r}\, e^{-\frac{1}{2L}\gamma^t\, t}\, v\,, \tag{C.4}$$

which can be shown to agree with the form in appendix E of [31].

The extremal solutions specified by (4.34)-(4.38) admit fake Killing spinors. To find them, we make the ansatz

$$\epsilon = \Big[f_+(y)P_+ + f_-(y)P_-\Big]\, u(t,\theta_i)\,, \tag{C.5}$$

where θ_i are the coordinates of the sphere S^3. This ansatz must satisfy the fake Killing spinor equations (4.1) (using (4.2)-(4.5)) when the solution data (4.34)-(4.38) is inserted. Analyzing the near-AdS limit of the equations, we find that the condition

$$i\gamma^t u = u\,. \tag{C.6}$$

is required. We expect that our solutions are at most half fake BPS, and we therefore impose the condition (C.6) when constructing exact fake Killing spinors. With this, it can be shown that the equations $\hat{\mathcal{D}}\epsilon = 0$ and $\mathcal{D}_y\epsilon = 0$ are solved if

$$f_\pm(y) = H(y)^{-\frac{1}{2+3k^2}}\,\Big[\sqrt{f(y)} \mp \sqrt{f(y)-1}\Big]^{1/2}\,, \tag{C.7}$$

where $f(y)$ is given in (4.35).

Next $\mathcal{D}_t\epsilon = 0$ reduces to

$$i\,\partial_t u = \frac{1}{2L}u\,, \tag{C.8}$$

with solution

$$u(t,\theta^i) = e^{-i\frac{1}{2L}t}\, v(\theta^i)\,. \tag{C.9}$$

The spinor v depends only on the S^3 coordinates. Finally, $\mathcal{D}_i\epsilon = 0$, with i running over the S^3 coordinates θ_i, simplifies to the conditions

$$P_\pm \left(\bar{\nabla}_i v \mp \frac{1}{2}\bar{\gamma}_i v \right) = 0\,, \tag{C.10}$$

which are simply equivalent to S^3 Killing spinor equations (C.3).

Thus our solutions admit fake Killing spinors

$$\epsilon = \Big[f_+(y)P_+ + f_-(y)P_- \Big]\, e^{-i\frac{1}{2L}t}\, v\,, \tag{C.11}$$

with $f_\pm$ given by (C.7) and with v, satisfying (C.3), a Killing spinor on the unit S^3 [30], constrained by the half fake BPS projection condition

$$i\gamma^t v = v\,. \tag{C.12}$$

Note that $f_\pm = g_\pm$ for $q = 0$, with $g_\pm$ in (C.2); in particular our fake Killing spinors asymptotically approach the AdS Killing spinors (C.1).

The Killing spinor bilinears $(\bar{\epsilon}\,\Gamma^\mu\epsilon)$ are Killing vectors of the bulk metric (4.34), whose isometry group is $\mathbb{R}\times \mathrm{SO}(4)$. This is the compact subgroup of the $\mathrm{SO}(4,2)$ group whose Killing vectors are denoted by $K^\mu_{[AB]}$ and whose spinor representation has generators $\gamma^{[AB]}$ given by

$$\gamma^{[ab]} = \frac{1}{2}\gamma^a\gamma^b \qquad a,b = 0,1,2,3,4 \qquad \text{rotations and boosts} \tag{C.13}$$

$$\gamma^{[a5]} = \frac{1}{2}\gamma^a \qquad \text{energy and “momentum”}. \tag{C.14}$$

The spinor bilinears for both pure AdS_5 and $\mathbb{R}\times S^3$ solutions are then given by

$$(\bar{\epsilon}_0\Gamma^\mu\epsilon_0) = \bar{v}\gamma^{[a5]}v\, K^\mu_{[a5]} + \frac{1}{2}\bar{v}\gamma^{[ab]}v\, K^\mu_{[ab]} \tag{C.15}$$

$$(\bar{\epsilon}\Gamma^\mu\epsilon) = \bar{v}\gamma^{[05]}v\, K^\mu_{[05]} + \frac{1}{2}\bar{v}\gamma^{[ab]}v\, K^\mu_{[ab]}, \qquad a,b = 1,2,3,4. \tag{C.16}$$

Note the restriction to energy and spatial rotations in the $\mathbb{R}\times S^3$ case which is due to the projection constraint (C.12).

References

[1] K. Skenderis and P.K. Townsend, *Gravitational stability and renormalization-group flow*, *Phys. Lett.* **B 468** (1999) 46 [hep-th/9909070].

[2] O. DeWolfe, D.Z. Freedman, S.S. Gubser and A. Karch, *Modeling the fifth dimension with scalars and gravity*, *Phys. Rev.* **D 62** (2000) 046008 [hep-th/9909134].

[3] J. de Boer, E.P. Verlinde and H.L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003 [hep-th/9912012];
J. de Boer, *The holographic renormalization group*, *Fortschr. Phys.* **49** (2001) 339 [hep-th/0101026];
A. Lawrence and A. Sever, *Holography and renormalization in Lorentzian signature*, *JHEP* **10** (2006) 013 [hep-th/0606022].

[4] P.D. Mannheim, *Brane-localized gravity*, World Scientific Press (2005).

[5] C. Csáki, J. Erlich, T.J. Hollowood and Y. Shirman, *Universal aspects of gravity localized on thick branes*, *Nucl. Phys.* **B 581** (2000) 309 [hep-th/0001033].

[6] D.Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, *Phys. Rev.* **D 69** (2004) 104027 [hep-th/0312055].

[7] K. Skenderis and P.K. Townsend, *Hamilton-Jacobi for domain walls and cosmologies*, *Phys. Rev.* **D 74** (2006) 125008 [hep-th/0609056].

[8] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of AdS_5 and its field theory dual*, *JHEP* **05** (2003) 072 [hep-th/0304129].

[9] A. Celi, A. Ceresole, G. Dall'Agata, A. Van Proeyen and M. Zagermann, *On the fakeness of fake supergravity*, *Phys. Rev.* **D 71** (2005) 045009 [hep-th/0410126].

[10] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett.* **B 115** (1982) 197.

[11] L.A.J. London, *Arbitrary dimensional cosmological multi-black holes*, *Nucl. Phys.* **B 434** (1995) 709.

[12] K. Behrndt, A.H. Chamseddine and W.A. Sabra, *BPS black holes in $N = 2$ five dimensional AdS supergravity*, *Phys. Lett.* **B 442** (1998) 97 [hep-th/9807187].

[13] K. Behrndt, M. Cvetič and W.A. Sabra, *Non-extreme black holes of five dimensional $N = 2$ AdS supergravity*, *Nucl. Phys.* **B 553** (1999) 317 [hep-th/9810227].

[14] R.C. Myers and O. Tafjord, *Superstars and giant gravitons*, *JHEP* **11** (2001) 009 [hep-th/0109127].

[15] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170]; *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys. Rev.* **D 60** (1999) 104026 [hep-th/9904197].

[16] M.J. Duff, *TASI Lectures on branes, black holes and anti-de Sitter space*, hep-th/9912164.

[17] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS_5 black holes*, *JHEP* **02** (2004) 006 [hep-th/0401042]; *General supersymmetric AdS_5 black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129];
H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS_5 black holes*, *JHEP* **04** (2006) 036 [hep-th/0601156].

[18] M. Cvetič et al., *Embedding AdS black holes in ten and eleven dimensions*, *Nucl. Phys.* **B 558** (1999) 96 [hep-th/9903214].

[19] C.J. Gao and S.N. Zhang, *Dilaton black holes in de Sitter or anti-de Sitter universe*, *Phys. Rev.* **D 70** (2004) 124019 [hep-th/0411104]; *Higher dimensional dilaton black holes with cosmological constant*, *Phys. Lett.* **B 605** (2005) 185 [hep-th/0411105].

[20] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017].

[21] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Novel local CFT and exact results on perturbations of $N = 4$ super Yang-Mills from AdS dynamics*, *JHEP* **12** (1998) 022 [hep-th/9810126].

JHEP12(2007)023

[22] J.P. Gauntlett and S. Pakis, *The geometry of $D = 11$ Killing spinors*, *JHEP* **04** (2003) 039 [hep-th/0212008].

[23] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[24] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Continuous distributions of D3-branes and gauged supergravity*, *JHEP* **07** (2000) 038 [hep-th/9906194].

[25] A. Brandhuber and K. Sfetsos, *Wilson loops from multicentre and rotating branes, mass gaps and phase structure in gauge theories*, *Adv. Theor. Math. Phys.* **3** (1999) 851 [hep-th/9906201].

[26] M. Cvetič, H. Lü, C.N. Pope, A. Sadrzadeh and T.A. Tran, *Consistent* SO(6) *reduction of type IIB supergravity on S^5*, *Nucl. Phys.* **B 586** (2000) 275 [hep-th/0003103].

[27] M. Kiermaier, private communication.

[28] S.S. Gubser, *Curvature singularities: the good, the bad and the naked*, *Adv. Theor. Math. Phys.* **4** (2002) 679 [hep-th/0002160].

[29] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[30] H. Lü, C.N. Pope and J. Rahmfeld, *A construction of Killing spinors on S^n*, *J. Math. Phys.* **40** (1999) 4518 [hep-th/9805151].

[31] M.C.N. Cheng and K. Skenderis, *Positivity of energy for asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 107 [hep-th/0506123].

[32] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121].

[33] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[34] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, hep-th/0404176.

[35] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [hep-th/0505190].

[36] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys.* **B 631** (2002) 159 [hep-th/0112119].

[37] M. Bianchi, O. DeWolfe, D.Z. Freedman and K. Pilch, *Anatomy of two holographic renormalization group flows*, *JHEP* **01** (2001) 021 [hep-th/0009156].

[38] G.W. Gibbons, S.W. Hawking, G.T. Horowitz and M.J. Perry, *Positive mass theorems for black holes*, *Commun. Math. Phys.* **88** (1983) 295.

[39] K. Skenderis and P.K. Townsend, *Hidden supersymmetry of domain walls and cosmologies*, *Phys. Rev. Lett.* **96** (2006) 191301 [hep-th/0602260].

[40] A. Ceresole and G. Dall'Agata, *Flow equations for non-BPS extremal black holes*, *JHEP* **03** (2007) 110 [hep-th/0702088].

[41] E. Bergshoeff et al., *$N = 2$ supergravity in five dimensions revisited*, *Class. and Quant. Grav.* **21** (2004) 3015 [*Erratum ibid.* **23** (2006) 7149] [hep-th/0403045].